THE RICE INSTITUTE PAMPHLET

Vol. XII April, 1925 No. 2

Published by
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A university of liberal and technical learning founded by William Marsh Rice in the City of Houston, Texas, and dedicated by him to the advancement of Letters, Science, and Art

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FOREWORD

DURING the second half of the year 1924, I had the honour of visiting a considerable number of American universities and scientific foundations, where I delivered lectures. These lectures, held under the auspices of the Commission for Relief in Belgium, were terminated at the Rice Institute in Houston. The administration of this notable scientific institution has kindly requested me to reproduce the essential part of them for translation and printing in its publication. It is, therefore, the condensed text of this series of lectures which will be found in the following pages.

CHARLES DE LA VALLÉE POUSSIN

ON THE APPROXIMATION OF FUNCTIONS OF A REAL VARIABLE AND ON QUASI-ANALYTIC FUNCTIONS [1]

I

SURVEY OF THE ORIGINS AND DEVELOPMENT OF THE THEORY OF APPROXIMATION [2]

1. *Weierstrass's first theorem. Remarks suggested by the title of the Note containing this theorem*

The recent investigations on the approximation of functions take their point of departure in a note of Weierstrass presented in 1885 to the Academy of Sciences at Berlin and containing two theorems of which this is the first: *Every function of x continuous in the interval (ab) is developable in a uniformly convergent series of polynomials in that interval.*

This is certainly a perfectly precise and extremely simple statement, of which the proof, as we shall see, is also quite simple. Yet the theorem seemed quite remarkable to the contemporaries of Weierstrass, and created considerable stir; and there is no doubt that it seemed quite remarkable

[1] Lectures delivered at the Rice Institute on December 16, 17, and 19, 1924, by Professor Charles de la Vallée Poussin of the University of Louvain. Translated from the French by Professor Griffith C. Evans of the Rice Institute.

[2] We have already treated analogous ideas in a lecture delivered at a meeting of the Swiss Mathematical Society, held at Fribourg, Feb. 24, 1918 (L'Enseignement Mathématique, t. XX, 1918, p. 23), and more recently, in a lecture given to the Accademia Romana dei Nuovi Lincei, at Rome, in 1923 (Lezioni publiche tenute nella Settimana Accademica, 26 aprile — 2 maggio 1923).

to Weierstrass himself. We find the proof of this in the very title which Weierstrass gave to his note, a title which excites questions which reach beyond the scope of the theorem which was proved. It runs as follows: *On the analytic representability of so-called arbitrary functions of a real variable.*[1] Is this really the same thing as the special proposition quoted above? The terms employed in this caption are so vague that it would seem that the first thing to do would be to find their precise meaning. What is meant, generally, by an "analytic representation"? What are the so-called "arbitrary" functions in question? What is this species of antinomy between the words "analytic" and "arbitrary" which Weierstrass believes he has dissolved? Indeed, the title selected by Weierstrass refers much more to contemporary concepts than to the real content of the memoir. We shall persuade ourselves of this by discovering what was understood at the time by "arbitrary function."

Previously, at the time of Euler, for instance, what one called a function was an algebraic expression. The simplest sort of function was a power of x, like x^2, x^3, . . . A function was thus defined by a certain process of calculation, a certain law which enabled one to proceed from the value of x to the value of the function, and this law was *the same* for all values of x. Afterwards were considered functions defined by more complicated formulas, for example, by a series of powers like the series of Maclaurin or Taylor; but mathematicians were persuaded that such a formula defined a unique law for proceeding from the value of the variable to that of the function. It was sufficient to know the function in an interval no matter how small in order to

[1] *Über die analytische Darstellbarkeit sogenannter willkürlicher Funktionen einer reellen Veränderlichen.* Berl. Ber. (1885) p. 633, p. 789.

deduce that law, and therefore, also, the knowledge of the function in an arbitrary interval.[1]

To this original concept of *analytic function* (that is, function given by a formula which binds its definitions for different values of x) is opposed the concept of *arbitrary function*, a concept in which precisely this liaison is suppressed. The definition of arbitrary function is *point wise;* it is made *point by point*, without there being any longer a dependence between the definitions of the function in two different points. It was natural to suppose that this *dissociation* of the function would, as a consequence, render its representation by one single formula impossible. Thus it was in early days that the two concepts of analytic and arbitrary functions were contrasted.

But the validity of this distinction was put in doubt in a remarkable manner, in the early part of the nineteenth century, by Fourier's researches on trigonometric series. In fact, Fourier showed that one could represent by a single trigonometric series, built up with a single set of coefficients, functions which were up to that time considered as different; e.g., $\sin x$ between 0 and π and $\cos x$ between π and 2π. Thus was set the problem of the analytic representation of so-called arbitrary functions.

Fourier's series did not completely solve the problem. Even a continuous function is expressible in a trigonometric series only by means of certain conditions which are not always satisfied. The problem of the analytic representation of a continuous function, given arbitrarily, was solved for the first time in the memoir in question, of Weierstrass; and this is the explanation of the somewhat ambitious title which the author gave to it.

[1] This is the property which is possessed by functions of a complex variable which we speak of now as *analytic*. We shall return to it in regard to quasi-analytic functions.

We may, nevertheless, point out that Weierstrass's theorem is not paradoxical at bottom. We are dealing in fact, only with continuous functions. But a continuous function is not completely dissociated; the function is completely determined by the values which it takes for rational values of x, and, therefore, by a denumerable infinity of conditions, since the values for x irrational may be inferred from the others in view of the continuity. Moreover, a series of polynomials contains a denumerable infinity of parameters. The infinity of parameters and the infinity of conditions are aggregates of the same power. Hence it is not surprising that such a series can represent any continuous function.

But in his caption Weierstrass does not say continuous; he merely says: *On the analytic representability of so-called arbitrary functions.* If then we take the title as it stands, we may properly ask what truth there may be in the statement of this general possibility.

The very special analytic representation which is used by Weierstrass is that by means of a series of polynomials. If further we suppose, with Weierstrass, that the series is uniformly convergent, it is an elementary theorem that that representation applies only to continuous functions.

Let us now discard the condition of uniformity of convergence and ask what will then be the functions which are developable in series of polynomials. The answer to this question is found in the fundamental investigations made by Baire. In his thesis (1899), Baire published a classification of functions of the highest importance. It runs as follows:

Continuous functions form the class 0; discontinuous functions which are limits of continuous functions form the class 1; the functions which are limits of functions of class 1 and are not of class 1, or class 0, are of class 2; and so on.

The functions which can be expressed in series of polynomials are those of classes 0 and 1, and no others. Baire in his thesis gave a celebrated theorem for the purpose of characterizing the functions of class 1, which bears his name,[1] and which is one of the most profound and fertile of the theory of functions of a real variable. Thanks to this theorem it is easy to define the functions which are not of class 1, and which, therefore, it is not possible to represent by a series of polynomials. Perhaps they will admit some other sort of analytic representation than this. But what then, speaking generally, do we mean by *analytic representation?*

Weierstrass would perhaps have been much embarrassed in replying to this question. The question is in fact rather vague. Nevertheless there is a paper, by Lebesgue, entitled: *On the functions which are representable analytically*,[2] where we can find the basis of an answer to the question. If we accept Lebesgue's point of view, and it would be difficult to do otherwise, the analytically representable functions are those which may be defined in terms of continuous functions by means of a denumerable infinity of processes of passing to the limit, — in other words, they are the ones which come under Baire's classification.

Now there exist, at least theoretically, functions which do not come under this classification. Thus it follows that the title of Weierstrass's memoir announces a possibility open to debate. Weierstrass is the one among mathematicians most concerned with rigour, and the one who has given the best models of it. Is it not, therefore, quite startling to see him put, as he did, such a precise and simple theorem under a somewhat ambiguous title, — one which leads to

[1] Baire's theorem: A function of class 1 is punctually discontinuous on every perfect set.

[2] Journal de Mathématiques, t. 60 (1905), p. 139.

the most thorny problems of mathematical philosophy, and moreover, one which, taken literally, would seem to be incorrect?

2. *Original proof of Weierstrass's theorem*

Let us turn now to the proof first given by Weierstrass for his theorem, reproduced by E. Picard in the first volume of his *Traité d'Analyse*, and become classic.

Weierstrass endeavors first, given a function $f(x)$ continuous in an interval ab, to construct a polynomial which comes arbitrarily near to the function in that interval. He takes his point of departure in the integral, familiar in the calculus of probabilities,

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}dt = 1.$$

By changing t into $t\sqrt{n}$, it becomes

$$\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} e^{-nt^2}dt = 1.$$

If we suppose that n is a positive integer increasing indefinitely, the only values of t which give an effective contribution to the integral are those infinitely near to zero. It follows that if α and β are two numbers of different signs, the integral

$$(1) \qquad \sqrt{\frac{n}{\pi}}\int_{\alpha}^{\beta} e^{-nt^2}dt$$

will have the same limit as the preceding when n tends to infinity, and this limit will therefore be unity.

Let now $f(x)$ be a function of x continuous in the interval (a, b), and x a value between a and b. We form the integral

$$(2) \qquad F(x) = \sqrt{\frac{n}{\pi}}\int_{a}^{b} f(t)e^{-n(x-t)^2}dt.$$

It is the same as

$$F(x) = \sqrt{\frac{n}{\pi}} \int_{a-x}^{b-x} f(t + x) e^{-nt^2} dt. \tag{3}$$

When n is allowed to become indefinitely great, we see, by comparison with (1), that the value of the integral (3) is a mean among the values taken by $f(x + t)$ as t ranges between the two limits of opposite sign $a - x$ and $b - x$. But since only the infinitely small values of t contribute anything, the mean value of $f(x + t)$ approaches $f(x)$ uniformly, the function being continuous. Hence $F(x)$ approaches $f(x)$ uniformly as n becomes infinite, through any interval interior to (a, b).

But the end in view is not yet attained, since $F(x)$ is not a polynomial; and this is in fact the objection to Weierstrass's method. In order to substitute a polynomial for $F(x)$ it is necessary to replace the exponential $e^{-n(t-x)^2}$, in the integral (2), by its uniformly convergent development in powers of $x - t$, and to keep in this development merely the number of terms necessary to get the desired approximation. By taking n sufficiently great, and taking in the development of the exponential a sufficient number of terms, we are able then to construct a polynomial which comes as near as we please to $f(x)$.

The problem of expressing $f(x)$ in a series of polynomials, and that of constructing a polynomial which approximates it to any desired degree are entirely equivalent. In fact, if P_n is a polynomial of degree n, and we have a sequence of polynomials $P_1, P_2, \ldots, P_n, \ldots$ approaching $f(x)$ uniformly as n becomes infinite, we also have an expression for $f(x)$ in a uniformly convergent series of polynomials, as follows:

$$f(x) = P_1 + (P_2 - P_1) + \ldots + (P_n - P_{n-1}) + \ldots$$

Hence Weierstrass's theorem is proved.

3. *Landau's integral*

The inconvenience of Weierstrass's method is that the function $F(x)$ defined by the integral (2) is not a polynomial in x. The properties of the integral which are used by Weierstrass relate to the presence of the exponential factor $\sqrt{n}\, e^{-n(x-t)^2}$, which has been called the *factor of discontinuity*. Hence it occurred to me to construct an integral similar to (2), but with another factor of discontinuity which should be a polynomial in x. Thus I was led to form an integral which since has received the name of *Landau's integral*, since Landau used it a few months before me for the same purpose.[1] However, this integral was considered even earlier by Hermite, and is found in his correspondence with Stieltjes, — a fact which Landau pointed out himself.

We replace the integral (1), which is the basis of Weierstrass's method, by the following,

$$\frac{k_n}{2}\int_{-1}^{+1}(1-t^2)^n dt = 1, \quad k_n = \frac{2\cdot 4 \ldots 2n}{1\cdot 3 \ldots 2n-1},$$

in which n is an integer. As before, when n increases indefinitely the only values of t important for the value of the integral are those infinitely near to zero.

Let now $f(x)$ be a function continuous in the interval (a, b). We shall assume this interval interior to $(0, 1)$. This hypothesis is indeed legitimate, since if it does not happen to be verified, we can satisfy it by a linear change of variable, which transforms a polynomial into another of the same degree.

With this understood, *Landau's integral* is the following:

$$P_{2n}(x) = \frac{k_n}{2}\int_0^1 f(t)\,[1-(x-t)^2]^n dt.$$

[1] *Über die approximation einer stetiger Funktion durch eine ganze rationale Funktion.* Rend. di Palermo, vol. 25 (1908), p. 337.

This can be substituted for the integral $F(x)$ of Weierstrass, for it yields exactly the same reasoning on the mean values. But it resolves precisely into a polynomial in x, $P_{2n}(x)$, of degree $2n$, which consequently approaches $f(x)$ as n becomes infinite. Thus the end is attained.

4. *Two problems studied by means of Landau's integral*

Landau's integral lends itself to the solution of interesting problems, of which I proposed to myself the two following: [1]

1. What is the order of approximation of $P_{2n}(x)$ to $f(x)$ relative to $1/n$ as n becomes infinite?
2. Are the successive derivatives of $f(x)$ represented approximately by those of the approximate polynomial $P_{2n}(x)$?

The first question, which Lebesgue [2] studied contemporaneously with me, is that of the *order of approximation*. It became the starting point of a theory which has since received great development, to which I will return in a few moments.

The second question, concerning the *differentiability* of the representation, was not new. Painlevé [3] had solved it in 1908 in the case of continuous derivatives. But, as I showed, Landau's integral furnishes a solution of the problem which is much better, and indeed almost perfect; the derivative of P_{2n} of any order p converges towards the derivative of the same order of $f(x)$ at the point x under the single condition of the existence of the latter derivative at

[1] *Sur l'approximation des fonctions de variables réelles et de leurs dérivées par des polynomes et des suites limitées de Fourier.* Bull. de l'Acad. Roy. de Belgique (Classe des Sciences), No. 3, Mars, 1908.

[2] *Sur la représentation approchée des fonctions.* Rend. di Palermo, vol. 26 (1908), p. 325.

[3] Comptes Rendus, t. 126 (1898), p. 459.

the point. That is quite the most remarkable property of Landau's integral, since as process of approximation of $f(x)$ it is only of mediocre value. In fact, the greatest value of Landau's integral for me was to suggest the construction of another integral which is its analogue for trigonometric approximation. This integral, analogous to Landau's, initiates investigations decidedly more interesting than Landau's integral itself. We may then naturally turn to the consideration of trigonometric approximation and the investigations which I have just mentioned.

5. *Trigonometric approximation and the analogue of Landau's integral*

The problem of trigonometric representation is as old as that of representation in terms of polynomials, and goes back to the original note of Weierstrass of 1885, that note containing a second theorem, as follows: *Every function, continuous and periodic, of period* 2π, *is developable in a uniformly convergent series of finite trigonometric expressions.*

A finite trigonometric expression of degree n is a polynomial of degree n in $\sin x$ and $\cos x$, or, what is the same thing, an expression of the form

$$\alpha_0 + (\alpha_1 \cos x + \beta_1 \sin x) + (\alpha_2 \cos 2x + \beta_2 \sin 2x) + \ldots + \ldots + (\alpha_n \cos nx + \beta_n \sin nx),$$

in which the coefficients α, β are constants.

It is well to notice that, whenever we are concerned with trigonometric approximation, the function $f(x)$ is always assumed to be *periodic*.

Weierstrass's second theorem has since received a large number of different demonstrations. But here I am only concerned with the one which I gave in 1908.[1] It is based

[1] Bull. Acad. Roy. de Belgique (Classe des Sciences), loc. cit.

on the consideration of an integral analogous to Landau's, possessing all of its advantages.

Let $f(x)$ be the continuous periodic function to be represented. The integral which defines a finite trigonometric expression $Q_n(x)$ of order n is the following:

$$Q_n(x) = l_n \int_0^{2\pi} f(t) \left(\cos \frac{x-t}{2}\right)^{2n} dt,$$

where the factor l_n is the reciprocal of the value of the well-known integral

$$\int_0^{2\pi} \cos^{2n} \frac{t}{2}\, dt = 4 \int_0^{\frac{\pi}{2}} \cos^{2n} t\, dt = \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n}\, 2\pi.$$

In this new integral $Q_n(x)$, the factor of discontinuity is $\cos^{2n} \frac{t-x}{2}$, which is a finite trigonometric expression of order n in x. We can treat this factor exactly in the same way as that which appears in Landau's integral, and we show in this way that $Q_n(x)$ converges to $f(x)$ when n becomes infinite, and possesses exactly the same properties as the polynomial $P_{2n}(x)$ of Landau with respect to the approximation and differentiability of the representation.

But what is most interesting in this new integral is that it leads, as I showed in my memoir of 1908, to the definition of a new method of summing divergent series.

6. *A new method of summing divergent series*

The trigonometric polynomial $Q_n(x)$, defined by the above integral, is connected in a very interesting way with the Fourier's series of $f(x)$. In fact, if we designate by a_k and b_k the Fourier constants for $f(x)$, to wit:

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx\, dx, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx\, dx,$$

the expression $Q_n(x)$ is resolved into the finite trigonometric sum of order n

$$\frac{1}{2}a_0 + \sum_{k=1}^{n} \frac{n(n-1)\dots(n-k+1)}{(n+1)(n+2)\dots(n+k)} (a_k \cos kx + b_k \sin kx).$$

Now the calculation of this sum and of its limit for $n = \infty$ constitutes a process of summation for the Fourier series. In fact, each term of this sum is obtained by multiplying the term of the same rank of the Fourier series by a numerical factor. These successive factors, for a given n, are continually decreasing, are < 1, and all approach 1 as n is let approach infinity. Since all these factors vanish from the $(n+1)^{th}$ on, this method yields the advantage of having a finite sum for each value of n. In accordance with what we have said, this process enables us to sum the Fourier series and its successive derivatives at a given point, under the single condition of the existence of these derivatives at the point.

These are the results to which I attained in 1908. Since then, several skilful geometers have concerned themselves with this process of summation and have shown that it has a power, importance and interest which I did not expect.

The process was applied by Plancherel to the summation of the series of Laplace and Legendre.[1] Kogbetliantz has applied it quite recently to ultraspherical functions.[2] Gronwall has made investigations even more searching. Some years ago he showed that this method of summation has at least all the generality of those by the Cesaro means.[3]

[1] *Sur l'application aux séries de Laplace du procédé de M. de la Vallée Poussin.* Comptes Rendus, t. 152 (1911), p. 1226.—Rend. di Palermo, vol. 33 (1912), p. 41.

[2] *Sur la sommation des fonctions ultrasphériques par la méthode* (Σ_0) *de M. de la Vallée Poussin.* Rend. di Palermo, vol. 46 (1922), p. 146.

[3] *Über einige Summationsmethoden und ihre Anwendung auf die Fouriersche Reihe.* Journal für die reine und angewandte Mathematik, Bd. 147 (1917), p. 16.

But I have learned, through a kind communication from Dr. Gronwall, that he has continued his researches in this direction, and has obtained most important results, which he has not yet published. With his permission, I indicate one of them which throws considerable light on the question.

A series $u_0 + u_1 + u_2 + \ldots$ is summable in my way and has for its sum the value s if the expression

$$V_n = \sum_{k=0}^{n} \frac{n!\,n!}{(n-k)!\,(n+k)!}\, u_n$$

tends towards s as n becomes infinite. Gronwall found the generating identity for the V_n. It is the following:

$$\sum_0^\infty u_n x^n = \sqrt{1-y} \sum_0^\infty \frac{(2n)!}{2^{2n} n!\, n!}\, V_n y^n, \text{ where } y = \frac{4x}{(1+x)^2}.$$

This striking identity directed Gronwall along the road to interesting generalizations of my method of summation. I cannot elaborate them here; but the identity exhibits directly an important fact, viz., that if V_n has a limit, $\Sigma u_n x^n$ necessarily approaches the same limit, as x approaches unity.[1]

If we apply this remark to the summation of Fourier's series, it follows that Poisson's method of summation must offer all the advantages of my own. I noticed the fact in 1908 and proved it in my Memoir, retracing all the proofs in detail. It is seen, thanks to Gronwall's elegant formula, that it was only the particular application of a general property.

7. *The problem of the order of approximation*

The most important of the problems which have been attacked in the study of approximation is that of the *order*

[1] We see in fact that we have

$$\frac{1}{\sqrt{1-y}} = \sum_0^\infty \frac{(2n)!}{2^{2n} n!\, n!}\, y^n.$$

of approximation. Let us define first what we mean by *approximation.* For example, let a continuous function $f(x)$ be represented by means of a polynomial of degree n, and let $P_n(x)$ be such a polynomial. The difference $f - P_n$ is the *error* of the approximation, and is a function of x; its maximum value in the interval of representation is the *approximation* ρ_n. This positive number approaches 0 as $1/n$ approaches zero, if the polynomial P_n is well chosen. It is therefore an infinitesimal of a certain order with respect to $1/n$. The problem of the order of approximation is the following: *To determine the relation which exists between the order of approximation* ρ_n, *which* $f(x)$ *may admit for a finite expression of order* n, *and the differential properties of the function.*

I offered myself the beginnings of an answer to this very problem in 1908, while studying the approximation given by Landau's integral. I showed also that the function $|x|$ admits an approximation to the order of $1/n$ by a polynomial of degree n, and I raised the question of deciding whether or not that was the order of the best possible approximation.[1] This definite question had much more importance for the development of the subject than had the few isolated results which I had obtained, because that question caused the writing of the two most important memoirs on the subject, one by D. Jackson and the other by S. Bernstein.

The problem can be set in two inverse formulations: *the direct problem,* the only one which I had attacked, has for its object to find the possible order of approximation in terms of the assumed properties of the function; *the inverse*

[1] *Sur la convergence des formules d'interpolation entre ordonnées équidistantes.* Bull. Ac. Roy. de Belgique (Classe des Sciences), 1104 (1908). This memoir ends with a note *Sur l'approximation par un polynome d'une fonction dont la derivée est à variation bornée,* the note which was the occasion of the question cited.

problem, more difficult, consists in repassing from a given supposedly possible order of approximation to the differential properties resulting for the function which is represented.

It is the memoir by D. Jackson [1] which answers most completely the direct question, and that of S. Bernstein [2] which answers most completely the inverse problem.

The results contained in these two memoirs constitute the essential matter of the volume of the Borel Collection which I had the privilege of publishing under the title: *Leçons sur l'approximation des fonctions de variables réelles.*[3] I combined the results obtained by the two authors above named, and filled them out in many points; I changed or simplified the proofs; but I contributed little in the way of new materials to the construction.

There will be found in that volume quite a number of results connected with the problem proposed, some concerning functions which possess merely a finite number of successive derivatives, others relative to functions which are indefinitely differentiable, or even analytic. It will suffice for me here to reproduce merely one of these results, — the one which makes apparent in most striking fashion the mutual dependence which exists between the order of approximation and the differential properties of the function. It is the subject of a theorem which I gave for the

[1] *Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationnale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung.* Inaugural Dissertation, Göttingen (1911). This memoir was crowned by the Academy of Sciences of Göttingen to which it had been presented in answer to a problem set by the Academy identical with mine.

[2] *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné.* Memoires publiés par la Classe des Sciences de l'Academie Royale de Belgique. Collec. in-4, 2e série, t. IV, 1912. Crowned memoir presented for a prize question set by the Class in 1911 at my instigation.

[3] Paris, Gauthier-Villars (1919).

first time in a meeting of the Swiss Mathematical Society at Fribourg in 1918,[1] and deals with the trigonometric representation of a periodic function $f(x)$.

If $f(x)$ admits a derivative of order r satisfying a Lipschitz condition of order α $(0 < \alpha < 1)$,[2] then whatever n, $f(x)$ admits a trigonometric representation of order n, with an approximation

$$\rho_n < \frac{M}{n^{r+\alpha}} \qquad (M \text{ const.}).$$

Conversely, if it is possible to satisfy this last condition for every n, $f(x)$ possesses a derivative of order r which satisfies a Lipschitz condition of order α.

This theorem deals with trigonometric approximation, but there is an analogous theorem for representation in terms of polynomials in an interval (a, b). In fact the two methods of representation lead each to the other, a fact which Bernstein brought out most clearly by his use of *trigonometric polynomials*.

We show in fact, following Bernstein, that the representation of $f(x)$ in terms of polynomials may be derived from its trigonometric representation. We can always assume that the interval of representation is $(-1, +1)$, since any other may be reduced to it by a linear transformation. We make then the transformation $x = \cos \phi$. We have then for the function $f(\cos \phi)$, which is periodic and an even function, the trigonometric development

$$f(\cos \phi) = a_0 + a_1 \cos \phi + \ldots + a_n \cos n\phi + \ldots$$

But $\cos n\phi$ is a polynomial of degree n in $\cos \phi$; and it is this polynomial $P_n(\cos \phi)$, which Bernstein calls a *trigonometric polynomial*. Hence if we return to the variable x,

[1] L'Enseignement Mathématique, t. 20 (1918), p. 23.

[2] That is to say, one can assign a constant A such that for all x and δ we have

$$|f^{(n)}(x+\delta) - f^{(n)}(x)| < A\delta^{\alpha}.$$

we have the redresentation of $f(x)$ in a series of trigonometric polynomials

$$f(x) = a_0 + a_1 P_1(x) + \ldots + a_n P_n(x) + \ldots$$

The transformation moreover applies directly to a finite sum as well as to a series, a fact which justifies our statement.

As we have already had occasion to notice, the theory of approximation has been the cause of the discovery of important theorems which retain a very considerable interest even outside that theory. One of the most remarkable in this way is a theorem about trigonometric expressions, discovered by Bernstein; it plays the essential rôle in the solution of the *inverse problem* mentioned above. It runs as follows:

If the absolute value of a trigonometric expression of order n does not exceed M, the absolute value of its derivative does not exceed nM, and, consequently, the absolute value of its p^{th} derivative does not exceed $n^p M$.

The algebraic character of this theorem was brought out by Marcel Riesz in 1914.[1] I also pointed out this algebraic character in my book, not being aware of Riesz's theorem at that time.

I have already given the most characteristic theorem on the order of approximation. It is interesting only for functions which possess merely a finite number of successive derivatives. In my book on approximation, I attacked the problem of obtaining results of similar precision for indefinitely differentiable functions, and in particular, for analytic functions. I was not able to solve it completely. I shall not describe these results here, because I am inclined to think that *for these cases* it is just as

[1] *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome.* Jahresberichte der deutschen Mathematiker-Vereinigung, Bd. 23 (1914), p. 354.

interesting to study the convergence of their Fourier series. This study, moreover, is made by similar methods, and will be the object of our second lecture.

8. *The best approximation*

Given a function $f(x)$ continuous in an interval (a, b), there exists a polynomial which furnishes the best possible approximation in this interval among all those of degree n and this polynomial is unique: it is called *the polynomial of minimum approximation* of degree n. This polynomial P_n is completely characterized by the fact that the difference $f - P_n$ takes on its maximum value $n + 2$ times, with alternation of sign. These results were given long ago by Tchebycheff,[1] and the proofs were made rigorous by Kirschberger and Borel.[2] Analogous theorems hold for the best trigonometric approximation of order n. The trigonometric expression of order n, S_n, which gives the best approximation is characterized by the fact that $f - S_n$ takes on its maximum value $2n + 2$ times, with alternation of sign, in its interval of periodicity. Certainly, Bernstein has succeeded in determining the polynomial of minimum approximation in some quite remarkable cases. But, in general, the exact determination of this polynomial is an inaccessible problem. There isn't any really practicable method for the approximate calculation of this polynomial; and of course there is the same state of affairs in the case of trigonometric approximation.

Nevertheless, it is quite important, when we consider an approximate expression of a certain order, to know if it gives a good approximation; that is to say, whether or

[1] *Sur les questions de minima qui se rattachent à la représentation approximative des fonctions.* Mémoires de l'Acad. Imp. des Sciences de St. Petersbourg, Sciences Math. et Phys., Série 6, t. VII, 1859. Collected Works, vol. I.

[2] *Leçons sur les fonctions de variables réelles*, Paris, Gauthier-Villars (1905).

not the approximation obtained differs too much from the best possible one. To decide this, it is necessary to know a lower limit to the best approximation. There are, as a matter of fact, quite a number of criteria which I developed in my *Leçons sur l'approximation.* I shall mention the three most characteristic. In order of date, the first belongs to Lebesgue, the second to myself, and the third to Bernstein.

Lebesgue's criterion [1] concerns trigonometric approximation. It is stated as follows: *If the sum of order n of the Fourier series for the periodic function $f(x)$ does not exceed $\phi(n)$, the best approximation of order n is not less than $A\phi(n)/\log n$, where A is a numerical constant which can be determined once for all.*

A second criterion, which I published some months later,[2] concerns polynomial approximation. It may be stated as follows: *If the polynomial Q_n of degree n is such that the difference $f - Q_n$ takes values of alternating sign in $n + 2$ consecutive points in the interval (a, b), the smallest in absolute value of these $n + 2$ values is a lower limit for the best approximation.*

Finally, Bernstein's criterion [3] deals again with trigonometric representation. It is connected with the well-known minimal property of the Fourier sums. This is the statement of it: *Let a_k, b_k be the Fourier constants for the continuous periodic function $f(x)$; the best approximation for this function by a trigonometric sum of order n is not less than the square root of*

$$\frac{1}{2}\sum_{n+1}^{\infty}(a_k^2 + b_k^2).$$

We shall see in the next lecture that this criterion is particularly interesting when we apply it to analytic functions.

[1] Ann. de la Fac. des Sciences de Toulouse, série 3, t. 1 (1910).
[2] Bull. Acad. Roy. de Belgique Classe des Sciences (1910).
[3] S. Bernstein (1912), loc. cit.

II

RELATIONS BETWEEN THE PROPERTIES OF A PERIODIC FUNCTION AND THE LAW OF DECREASE OF ITS FOURIER COEFFICIENTS

1. *Condition for the unlimited differentiability of a periodic function*

Let $f(x)$ be a continuous function of period 2π, indefinitely differentiable. The condition of periodicity, we must observe, is absolutely indispensable for what follows. Such a function is developable as a Fourier series in the form

$$f(x) = \frac{1}{2} a_0 + \sum_1^\infty (a_k \cos kx + b_k \sin kx). \tag{1}$$

Its coefficients a_k and b_k are expressed by the well-known integrals

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos kx\, dx, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin kx\, dx.$$

Let us operate on the first of these by p successive integrations by parts, noticing that the terms at the limits drop out on account of the periodicity. Hence

$$a_k = \pm \frac{1}{\pi}\frac{1}{k^p}\int_0^{2\pi} f^{(p)}(x) \begin{pmatrix} \cos kx \\ \sin kx \end{pmatrix} dx$$

so that, if we designate by M_p the absolute maximum of the derivative of order p, we have, by the theorem of the mean,

$$|a_k| < \frac{2M_p}{k} \quad \text{and, in the same way,} \quad |b_k| < \frac{2M_p}{k^p}.$$

In these inequalities, the integer p is as large as we like; they prove then that, as k becomes infinite, the Fourier

coefficients a_k and b_k become infinitely small with respect to every negative power of k.

Conversely, if this condition is fulfilled, the series (1) and all its derivatives are uniformly convergent, so that the differentiation of the series and of its derivatives one after the other is permissible, and the periodic function $f(x)$ is indefinitely differentiable.

Thus, the necessary and sufficient condition in order that a continuous periodic function be indefinitely differentiable is that its Fourier coefficients a_k and b_k be infinitely small with respect to every negative power of k, or again, what amounts to the same thing, that:

$$\log\left|\frac{1}{a_k}\right|,\ \log\left|\frac{1}{b_k}\right|$$

be infinitely large in comparison with $\log k$. It may be observed that we have proved that a continuous periodic function which is indefinitely differentiable may be represented by a Fourier series which is indefinitely differentiable.

2. *Condition for analyticity*

We shall now look for the law of decrease to be imposed on the Fourier coefficients in order that the periodic function $f(x)$, which is real on the axis Ox and of period 2π, shall be analytic in every point of the axis. In this case, $f(x)$ is the value on Ox of an analytic function $f(z)$ of the complex variable $z = x + yi$, and every point of Ox is a regular point of $f(z)$. Consequently we can draw two lines $y = \pm b$, parallel to the axis and at equal distances above and below it, such that $f(z)$ has no critical point either in the infinite strip contained between these lines or on the lines themselves.

Under these conditions, we shall show that the Fourier

development of $f(z)$ is equivalent to a development in terms of a Laurent series.

Make the change of variables

$$t = e^{zi}$$

by means of which we have

$$f(z) = \phi(t).$$

If $f(z)$ were not periodic, $\phi(t)$ would not be a uniform function of t, because t is a periodic function of z. But by hypothesis, $f(z)$ admits the same period 2π as t itself, and in such case, $\phi(t)$ is a uniform function of t. Let us find the domain in the plane (t) which corresponds to the above strip in the plane (z).

If z traces out the line $y = b$, the modulus of t is constant and equal to e^{-b}; it is equal to e^{b} if z traces out the line $y = -b$. Hence, to the strip in the plane (z) corresponds, in the plane (t), the circular ring contained between the two circles with center at the origin and radii e^{-b} and e^{b}. The function $\phi(t)$, being holomorphic in this annular region and on the two circles, is developable in a Laurent series, which is convergent in the annular region and on the two circles, of the form

$$\phi(t) = A_0 + \sum_1^\infty A_k t^k + \sum_1^\infty \frac{B_k}{t^k}.$$

These two series converge on the two circles, and therefore when $t = e^{\pm b}$ their general terms approach 0 as k becomes infinite. It follows that $A_k e^{kb}$ and $B_k e^{kb}$ approach 0 as k becomes infinite.

If now we return to the variable z the Laurent development for $\phi(t)$ becomes the following development for $f(z)$:

$$\begin{aligned} f(z) &= A_0 + \Sigma\,(A_k e^{kzi} + B_k e^{-kzi}) \\ &= A_0 + \Sigma\,[(A_k + B_k)\cos kz + i(A_k - B_k)\sin kz] \end{aligned}$$

which is precisely the Fourier development for $f(z)$. Hence

the Fourier constants for $f(x)$ are

$$a_k = A_k + B_k,\ b_k = i(A_k - B_k).$$

It follows from this that the products $a_k e^{bk}$ and $b_k e^{bk}$ approach zero as k becomes infinite.

Conversely, if this condition is satisfied the Laurent development for $\phi(t)$ converges in the annular region, the Fourier development converges in the strip, and $f(z)$ is holomorphic in the strip.

Thus, the necessary and sufficient condition that $f(x)$, of period 2π, be analytic at every point on the axis of x is that a positive number b may be assigned such that the products $a_k e^{kb}$ and $b_k e^{kb}$ have the limit zero as k becomes infinite. That amounts to saying that a_k and b_k approach zero at least as rapidly as the terms of a convergent geometric progression; or, also, that: $\log\left|\frac{1}{a_k}\right|$, $\log\left|\frac{1}{b_k}\right|$ be infinitely great of order at least equal to k.

3. *Comparison of the preceding results*

If we compare the conclusions in the two preceding sections, we notice that the condition for the analyticity of $f(x)$ on the real axis is much narrower than that for unlimited differentiability, because the variable k increases much more rapidly than its logarithm. Consequently the class of analytic functions must be a very particular class of those which are indefinitely differentiable. There must be, then, between the two classes a considerable margin, which opens a place for intermediate classes. It is in this margin, as we shall see in the last lecture, that we must place the class of *quasi-analytic functions.*

We now proceed to study in more detail the law of decrease of the Fourier coefficients of a function by making

it correspond to the more special properties of the function to be developed. We commence with the infinitely differentiable functions.

4. *Indefinitely differentiable functions. Relation between the law of limitation for the successive derivatives and the law of limitation for the Fourier coefficients*

Consider a function $f(x)$, periodic of period 2π and infinitely differentiable. We shall assume, in order to simplify the writing (although it is in no wise essential), that the function is even and can be developed in a series of cosines alone, in the form

$$f(x) = \Sigma a_k \cos kx.$$

Assume that the successive derivatives of $f(x)$ are bounded according to the law

$$\sqrt[p]{|f^{(p)}|} < \phi(p) \tag{1}$$

where $\phi(p)$ is a positive non-decreasing function.

We have first the following theorem: If the function $\phi(p)$ is bounded, remaining less than an integer n, the function $f(x)$ is a finite trigonometric sum of order less than n.

In fact, we make (as before) p integrations by parts, and use the above given law of limitation, so that we obtain

$$|a_k| = \frac{1}{\pi k^p}\left|\int_0^{2\pi} f^{(p)}(x)\begin{pmatrix}\cos kx\\ \sin kx\end{pmatrix} dx\right| < 2\left(\frac{\phi(p)}{k}\right)^p. \tag{2}$$

We can let p become infinite in this formula. Hence if $\phi(p)$ remained less than n, this formula would give $|a_k| = 0$ as soon as k had become $\geq n$; and the Fourier development would reduce to a finite number of terms.

We may, therefore, from now on discard this case, at present of no interest to us, and assume that $\phi(p)$ becomes infinite with p. Without loss of generality we may assume

that $\phi(p)$ not only is non-decreasing, but actually increases with p.

With this understood, let us assign arbitrarily a number $m > 1$, and afterwards determine the integer p by the conditions

$$\phi(p) \leq \frac{k}{m} < \phi(p+1).$$

We denote by ψ the function inverse to ϕ; thus the two inequalities just given become the following,

$$p \leq \psi\left(\frac{k}{m}\right) < p+1.$$

Formula (2) then yields *a fortiori*

$$|a_k| < 2m\left(\frac{1}{m}\right)^{\psi\left(\frac{k}{m}\right)} = 2me^{-\psi\left(\frac{k}{m}\right)\log m} \tag{3}$$

We shall use this relation shortly. It can be simplified by making $m = e$, so that we have, in particular,

$$|a_k| < 2e \cdot e^{-\psi\left(\frac{k}{e}\right)} \tag{3'}$$

Thus to the law of limitation (1), on the successive derivatives of the function, corresponds the law of limitation (3′) on the Fourier coefficients, which is obtained by inversion of the function ϕ.

Let us turn now to the converse. Assume that we are given the law of limitation of the Fourier coefficients in the form

$$|a_k| < Ae^{-\psi(k)} \tag{4}$$

where A is a constant and $\psi(k)$ is a non-decreasing function tending to infinity as k becomes infinite. Let us see if we can deduce a law of limitation for the successive derivatives of the function by inverting the function ψ.

We shall in fact obtain such a law, but by means of a hypothesis on the rapidity of increase of $\psi(k)$. We

shall assume that ψ has a derivative and that a positive number α can be assigned sufficiently small so that x^α increases less rapidly than $\psi(x)$, or, so that $\psi(x)/x^\alpha$ is continually increasing. The logarithmic derivative of this quantity is not negative; hence we can assign a positive number α so that everywhere

$$\frac{x\psi'(x)}{\psi(x)} > \alpha. \tag{5}$$

Let us make this hypothesis. We have the development

$$f(x) = \Sigma a_k \cos kx;$$

consequently, by differentiating p times,

$$|f^{(p)}| < \Sigma k^p |a_k| < A\Sigma k^p e^{-\psi(k)} = A\Sigma \left(k^p e^{-\frac{\psi(k)}{2}}\right) e^{-\frac{\psi(k)}{2}}.$$

The positive series which has $e^{-\frac{\psi(k)}{2}}$ for its general term is convergent. The product of the sum of this series by A is a constant which we may designate by B^p; hence

$$|f^{(p)}| < B^p \max k^p e^{-\frac{\psi(k)}{2}}.$$

This maximum refers to the variation of k. It is reached for a value of k certainly less than that which makes the logarithmic derivative of the function negative. This derivative has the value

$$\frac{p}{k} - \frac{\psi'(k)}{2},$$

and has the same sign as

$$2p - k\psi'(k) = 2p - \frac{k\psi'(k)}{\psi(k)}\psi(k) < 2p - \alpha\psi(k).$$

It will therefore be negative if we have

$$\psi(k) = \frac{2p}{\alpha},$$

that is, if we have

$$k = \phi\left(\frac{2p}{\alpha}\right)$$

where ϕ is the inverse function of ψ. Hence the maximum will be attained for a value of k less than this value, and we shall have *a fortiori*

$$\sqrt[p]{|f^{(p)}|} < B\phi\left(\frac{2p}{\alpha}\right). \tag{6}$$

Accordingly, assuming the condition (5), there corresponds to every law of limitation of the Fourier coefficients given by (4) a law of limitation of the successive derivatives, which is given by inversion of the function ψ. This inversion will be indispensable to us in our study of *quasi-analytic* functions.

5. *Continuation of the preceding*

When the law of limitation of the Fourier coefficients is assumed to be of the form

$$|a_k| < Ae^{-\psi(k)}$$

as in the preceding section, we had to impose a supplementary condition on the function $\psi(k)$, in order to deduce by an inversion the law of limitation for the successive derivatives of the function $f(x)$. We may ask if the employment of a restriction was necessary.

It is not difficult to verify this necessity. In fact, a function which is indefinitely differentiable satisfies by that very fact a law of decrease of its Fourier coefficients, and this law can be written in the given form. If it were possible, in general, to deduce from that a law of limitation for the successive derivatives by inversion of the function $\psi(k)$, the simple fact of their existence would necessitate a law of limitation on them, assignable *a priori*. The contrary is not evident; nevertheless it is the contrary which is correct. In fact, Borel proved in his thesis that a function could be constructed of which all the derivatives would

exist, and would at a given point take on values given in an arbitrary manner. Moreover the formulas already given enable us to construct a function whose derivatives will not satisfy a law of limitation, given in advance, of the type

$$\sqrt[p]{|f^{(p)}|} < \phi(p).$$

In fact, this formula implies as a consequence the condition (3) of the preceding section. Now in this condition we can make $m = \sqrt{k}$, and obtain

$$|a_k| < 2\sqrt{k}\, e^{-\frac{1}{2}\psi(\sqrt{k})\log k} = 2\, k^{\frac{1}{2}-\frac{1}{2}\psi(\sqrt{k})}.$$

The second member is infinitely small in comparison with every negative power of k, for $\psi(\sqrt{k})$ increases indefinitely with k. Hence the function defined by the series

$$\Sigma\, k^{\frac{1}{2}-\frac{1}{2}\psi(\sqrt{k})}\cos kx$$

is indefinitely differentiable and its successive derivatives are not subject to the law of limitation given in the preceding section.

The introduction of a restrictive condition was really indispensable. In order to pass from the limitation of the a_k to that of the $f^{(p)}$ in the general case, it is necessary to introduce the order of infinity of $\log |1/a_k|$ with respect to $\log k$. For that purpose we must write the formula of limitation of the a_k in the form

$$|a_k| < Ae^{-\psi(k)\log k} = Ak^{-\psi(k)}. \tag{7}$$

In this case, the law of limitation of the $f^{(p)}$ is obtained without exception by inversion of the function ψ.

We can, in fact, write

$$|f^{(p)}| < \Sigma\, |k^p a_k| < A\,\Sigma\, k^{p-\psi(k)}.$$

Choose at pleasure a number α, $0 < \alpha < 1$. We have

$$|f^{(p)}| < A\,\Sigma\, k^{p-\alpha\psi(k)} k^{-(1-\alpha)\psi(k)}.$$

Now the series of which the general term is $k^{-(1-\alpha)\psi(k)}$ is convergent, and its product by A will not surpass a number

which we may designate by B^p. We have then

$$|f^{(p)}| < B^p \max k^{p-\alpha\psi(k)}.$$

This maximum is attained for a value of k less than that which makes the exponent negative, therefore less than $\phi(p/\alpha)$, where ϕ is the function inverse to ψ. Consequently we have, *a fortiori*

$$\sqrt[p]{|f^{(p)}|} < B\phi\left(\frac{p}{\alpha}\right). \tag{8}$$

Hence if we write the formula of limitation of the a_k in the form (7), the limitation of the derivatives is obtained, in every case, by the formula (8); that is by inversion of the function ψ.

6. *Entire analytic functions*

We return now to the analytic functions, and we shall study more closely the manner in which the law of decrease of the Fourier coefficients is connected with the properties of the function. All that we have previously said about indefinitely differentiable functions applies in particular to analytic functions; but analytic functions possess peculiar properties of their own, to which the law of decrease which interests us is directly related.

The investigation which we made in Section 2, already has placed in evidence the fact that if the periodic function $f(z)$ has critical points, the decrease of its Fourier coefficients depends primarily on the least distance between the real axis and a critical point, because it is on this distance that the number b depends, which we introduced into the formulas of Section 2.

We shall begin, however, by the investigation of functions which are entire, and have no critical point in the whole plane. In this case, as we shall show, the law of decrease of the Fourier coefficients is connected very closely with the

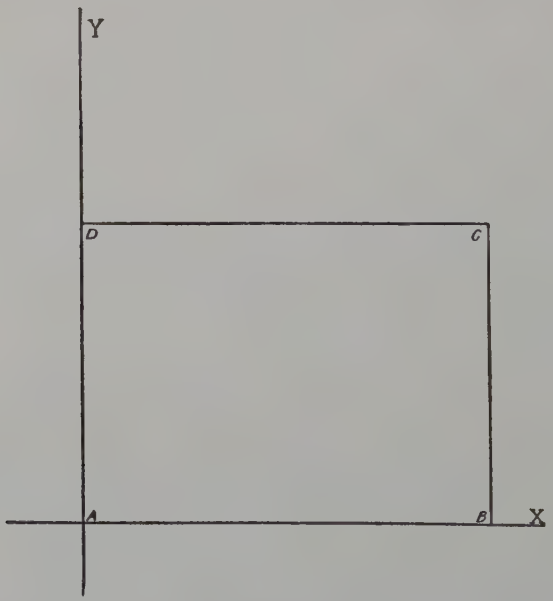
order of magnitude of the function as we recede indefinitely from the real axis.

Let $f(z)$ be a function of the complex variable $z = x + yi$, assumed to be periodic of period 2π, real on the real axis, and analytic over the whole plane.

Both of its Fourier coefficients a_k and b_k may be determined simultaneously by the single formula

$$a_k + ib_k = \frac{1}{\pi}\int_0^{2\pi} f(x)e^{kxi}dx.$$

Designate by AB the segment $(0, 2\pi)$ of the real axis (on which the above integration has been performed). On this segment as base construct the rectangle $ABCDA$ of



altitude y. Since the function $f(z)e^{kzi}$ is holomorphic in this rectangle, the integral on the side AB is the same as that on the contour $ADCB$, formed by the other three sides. But the integrals on the vertical sides AD and CB

cancel, on account of the periodicity of $f(z)$; so that the integral on AB is equal to that on DC. In other words, we can replace x by $x + yi$ in the integral written above. We have therefore

$$(1) \qquad a_k + ib_k = \frac{1}{\pi} e^{-ky} \int_0^{2\pi} f(x + yi) e^{kxi} dx.$$

From this we infer the inequality

$$(2) \qquad \sqrt{a_k^2 + b_k^2} < 2\, e^{-ky} \max |f(x + iy)|$$

where the maximum refers to the variation of x. We could deduce immediately from this formula the results already obtained in Section 2, in another way, with respect to the decrease of the coefficients a_k and b_k.

Since the function $f(z)$ is supposed to be entire, the positive number y in the formula (2) is arbitrary. We must choose it then so as to minimize the second member. This minimum depends upon the law of increase with y of the modulus of $f(x + yi)$. Write then

$$(3) \qquad |f(x + yi)| < Ae^{y\phi(y)}$$

where A is a constant and $\phi(y)$ a non-decreasing function of y.

We have first the following theorem, which is nothing but a particular form of the generalized theorem of Liouville: If, in (3), the function $\phi(y)$ is bounded and remains less or equal to an integer n for all y, the function $f(z)$ is a finite trigonometric sum of order n.

In fact, under this hypothesis, the relation (2) gives

$$\sqrt{a_k^2 + b_k^2} < 2\, Ae^{(n-k)y}.$$

If we let y increase indefinitely we see that a_k and b_k are zero if k is $> n$, which proves the theorem.

We shall assume then that $f(z)$ is not a finite trigonometric

sum and that $\phi(y)$, in (3), is a constantly increasing function which becomes indefinitely great with y.

The relation (2), if we take account of (3), becomes

$$\sqrt{a_k^2 + b_k^2} < 2Ae^{-[k-\phi(y)]y}$$

Designate by ψ the function inverse to ϕ; select arbitrarily a number α contained between 0 and 1; and then choose y by the condition

$$\phi(y) = \alpha k \quad \text{whence} \quad y = \psi(\alpha k).$$

We have

$$\sqrt{a_k^2 + b_k^2} < 2Ae^{-(1-\alpha)k\psi(\alpha k)}. \tag{4}$$

Hence we have found the law of limitation of the a_k, b_k by inversion of the function ϕ.

Let us turn to the reciprocal relation. Let the law of limitation of the Fourier coefficients be of the form

$$\sqrt{a_k^2 + b_k^2} < Ae^{-k\psi(k)}, \tag{5}$$

where A is a constant and $\psi(k)$ a constantly increasing function of k. We shall assume that $\psi(k)$ becomes infinite with k, since otherwise the function would not be an entire function. According to what we have seen in Section 2, it would unavoidably possess a critical point of which the ordinate would be equal to the upper bound of $\psi(k)$.

The development of $f(z)$ is the following:

$$f(z) = \sum_k (a_k \cos kz + b_k \sin kz).$$

If we designate in general by α_k a real quantity, the development may be put in the form

$$f(z) = \sum_k \sqrt{a_k^2 + b_k^2} \cos(kz + \alpha_k).$$

Hence

$$|f(x+yi)| < \sum_k \sqrt{a_k^2 + b_k^2} \frac{e^{ky} + e^{-ky}}{2} < \sum_k \sqrt{a_k^2 + b_k^2}\, e^{ky}.$$

And therefore, by means of condition (5), we find

$$|f(x+yi)| < A \sum_k e^{k(y-\psi(k))}.$$

Take again a number α, arbitrarily small. When this has been done, separate the preceding sum into two parts. The first part is to extend over those values of k for which $\psi(k)$ is $< y + \alpha$, that is, over the values $k < \phi(y + \alpha)$, where ϕ is the inverse function to ψ. The second part is to extend over the other values. We have then *a fortiori*

$$|f(x+yi)| < A \sum_0^{\phi(y+\alpha)} e^{ky} + \sum_{\phi(y+\alpha)}^{\infty} e^{-k\alpha}.$$

The first sum is an increasing geometric progression, and of the order of magnitude of its last term; the second is a decreasing geometric progression and has a finite value: we can therefore assign it an upper bound which does not depend on α. We can therefore assign a constant B independent of y such that

$$|f(x+yi)| < Be^{y\phi(y+\alpha)}, \tag{6}$$

the constant B possibly depending on the choice of α.

In the formula (5) we have accordingly a limitation of the modulus of $f(z)$, given by (6), — that is, by inversion of the function ψ.

7. *Digression. Study of the function $\phi(y)$ which is related to the law of increase of the modulus of an entire periodic function*

We consider an entire function $f(z)$, which is not a mere constant. We assume it to be periodic with period 2π. Moreover, we suppose it to be real on the real axis; and therefore at conjugate points it takes on conjugate values, of the same modulus. We shall consider the maximum of the modulus of $f(z)$ on a parallel to the real axis, of

ordinate y, positive; and we shall define the function $\phi(y)$, of y real and positive, by writing

$$\max |f(x+yi)| = Ae^{\nu\phi(\nu)}, \tag{1}$$

where A is a positive constant, chosen arbitrarily.

We commence by making an important observation. Writing, for simplicity

$$R(y) = max\,|f(x+yi)|$$

we shall show that $R(y)$ possesses a continuous derivative except possibly in isolated points.

In fact, if $|f(x+yi)|$ reaches its maximum only for a single value x_1 of x in the period, we shall have $\frac{\partial}{\partial x}|f(x_1+yi)| = 0$ in this point; hence necessarily

$$\frac{dR(y)}{dy} = \frac{\partial}{\partial y}|f(x_1+yi)|.$$

In this case the derivative exists and is continuous in the neighborhood of the point y.

On the other hand, if the maximum of $|f(x+yi)|$ is reached for several different values $x_1, x_2, \ldots$ of x in the period, the function $R(y)$ will have different derivatives in the directions of increasing and decreasing y. The former will be equal to the greatest and the latter to the least of the partial derivatives

$$\frac{\partial}{\partial y}|f(x_1+yi)|, \quad \frac{\partial}{\partial y}|f(x_2+yi)|, \ldots$$

Hence, when y passes through one of these exceptional values, the derivative of $R(y)$ will increase in value by a sudden jump.

The same conclusions apply accordingly to the function $\phi(y)$. It possesses in general a continuous derivative $\phi'(y)$ except possibly for particular values of y where there are two different derivatives $\phi'(y+0)$ and $\phi'(y-0)$, and the

upper derivative is greater than the lower. These two derivatives are respectively the limits of $\phi'(y \pm \epsilon)$ as ϵ approaches 0.

We know that the modulus of an entire function cannot attain its maximum in a closed region except on the frontier of the region. From this it follows easily that $y\phi(y)$ increases with y (supposed positive).

Consider in fact, the rectangle $ABCD$ of altitude y which has for base the segment AB of length equal to the period 2π. The maximum of the modulus of $f(z)$ in this rectangle cannot be attained on AB, because that is an axis of symmetry for the modulus, nor can it be attained on the vertical sides, on account of the periodicity. Hence the maximum cannot be reached except on CD. As we let y increase, the rectangle $ABCD$ enlarges; hence the maximum of the modulus of $f(z)$ in the rectangle is increased, and consequently the function $y\phi(y)$ increases with y.

This first result is almost immediate. The second is a little less so.

The derivative of $y\phi(y)$, *that is, the expression* $\phi(y) + y\phi'(y)$ *is also a constantly increasing function of* y, *positive.*

In fact, consider the entire function

$$F(z) = f(z)e^{kzi}$$

where k is a positive number, integer or not. Since the modulus of the exponential is constant on a parallel to the real axis, the modulus of $F(z)$ admits the period 2π. Hence if we consider a rectangle $ABCD$ of which the base AB is a horizontal segment of length 2π, the modulus of $F(z)$ cannot attain its maximum in the rectangle $ABCD$ except on the horizontal sides.[1] It follows evidently that the maximum of the modulus of $F(z)$ on the horizontal of

[1] We observe that since $F(z)$ is no longer real on Ox, that axis is no longer an axis of symmetry for the modulus.

ordinate y is a function of y which cannot have a maximum (for otherwise the modulus of F would also reach a maximum). This function of y has the form

$$Ae^{y[\phi(y)-k]}.$$

The derivative of the exponent is

$$\phi(y) + y\phi'(y) - k.$$

If $\phi(y) + y\phi'(y)$ did not increase as y passed through a value y_0, we could take the constant k equal to $\phi(y_0) + y_0\phi'(y_0)$; then the derivative just given would change from positive to negative or would stay zero. Hence the modulus of $F(z)$ would attain a maximum value on the line $y = y_0$, which is impossible.

From this we deduce the following theorem:

The function $\phi(y)$ cannot pass through a maximum when y varies from 0 to infinity.

In fact, if $\phi(y)$ should have a maximum for $y = y_0$, $\phi'(y_0 - 0)$ would be zero or positive at y_0 and negative at $y_0 + \epsilon$. Hence the function $\phi + y\phi'$ would change from the value $\phi(y_0) + y_0\phi'(y_0 - 0) \geq \phi(y_0)$ to the value

$$\phi(y_0 + \epsilon) + (y_0 + \epsilon)\phi'(y_0 + \epsilon) < \phi(y_0 + \epsilon) < \phi(y_0),$$

which contradicts the previous result.

If the function $f(z)$ is not a finite trigonometric expression, we know that $\phi(y)$ increases indefinitely with y. In this case, consequently, the function $\phi(y)$ may start by decreasing and reaching a minimum; but after this point it becomes definitely increasing and tends towards infinity.

We are going to make still more definite the properties of the function $\phi(y)$ by choosing the constant A in such a way that $\phi(y)$ remains finite for $y = 0$. For that purpose it is necessary to assign to A the maximum value of the modulus of $f(z)$ on the real axis. The function $\phi(y)$ is defined in this case by the formula

$$\max |f(x+yi)| = \max |f(x)| e^{y\phi(y)}. \tag{2}$$

We have then this theorem:

With this definition the function $\phi(y)$ cannot have a minimum.

In fact, if $\phi(y)$ had a minimum at y_0, $\phi(y)$ would be decreasing between 0 and y_0, $\phi'(y_0-0)$ would be zero, and the function $\phi(y) + y\phi'(y)$ would not be increasing between 0 and y_0, since it would pass from the value $\phi(0)$ to the smaller value $\phi(y_0) + y_0\phi'(y_0-0) \geq \phi(y_0)$.

From this we have the following conclusion:

If the function $\phi(y)$ is defined by the formula (2), *it is necessarily increasing as y varies from* 0 *to* ∞.

This function has neither maximum nor minimum; it varies then continually in the same sense. It tends to infinity, if $f(z)$ is not a finite trigonometric expression. In this case it is then evidently increasing. The conclusion is less immediate if $f(z)$ is a finite trigonometric sum, but it holds nevertheless, because $\phi(y)$ will be certainly increasing for large values of y. We shall verify this statement. For this purpose, let us express $f(z)$ in terms of imaginary exponential functions. If the order is n, we shall have

$$f(z) = a_0 e^{-nzi} + \dots$$

The term just written is the principal one for $y = \infty$. The value of A which is equal to the maximum of the modulus of $f(x)$ will be greater than the modulus of a_0, because we can choose x so that the sum of the unwritten terms will have the same argument as the first term.[1] We

[1] If we put $e^{xi} = x$, we may write

$$f(z) = x^{-n}(a_0 + a_1 x + \dots + a_{2n}x^{2n}).$$

Since the expression $a_1x + \dots + a_{2n}x^{2n}$ has the root 0, its argument is increased by at least one circumference when x describes the circle of radius 1 around the origin, and it can therefore be given the argument of a_0.

have asymptotically

$$\phi(y) = n + \frac{1}{y} \log \left| \frac{a_0}{A} \right|.$$

This logarithm is negative. Hence $\phi(y)$ approaches n by increasing, as y varies from 0 to ∞.

8. *Functions with critical points*

If the function $f(z)$, which is periodic and analytic on the x axis has critical points outside of that axis, the law of decrease of its Fourier coefficients depends, in the first instance, on the distance from the axis of the nearest critical point. That is the immediate consequence of the considerations which we have already made. (Section 2.)

Let b be the shortest distance; however small $\epsilon > 0$ is given, the Fourier coefficients a_k and b_k, as k becomes infinite, will become infinitely small with regard to the exponential $e^{-(b-\epsilon)k}$, while they cannot behave that way with regard to $e^{-(b+\epsilon)k}$. The order of magnitude of the Fourier coefficients in the scale of exponential functions is sufficient to determine the distance to the real axis of the nearest critical point.

In many cases, we can determine the law of decrease of the Fourier coefficients more definitely by specifying the nature of the critical point or points nearest the real axis. In my lessons on approximation (Chapters VIII and IX) I have indicated various cases in which it is possible to find the asymptotic values of the Fourier coefficients, the simplest case being that of polar singularities. It seems therefore unnecessary to discuss this question at present.

9. *On the order of approximation furnished by the Fourier sums of an analytic function*

When we develop an analytic function in a Fourier series, the Fourier sums furnish in general a very satisfactory

approximation. Practically speaking, this approximation is almost always of the order of the best possible. To put the matter exactly, we can in fact state the following proposition:

If, as n tends to infinity in an arbitrary manner, the approximation furnished by the Fourier sum of order n, for an analytic function $f(x)$, is an infinitely small quantity of a definite order with respect to $1/n$, this order is also that of the best possible approximation.

This statement is evidently a consequence of the following theorem:

There are an infinite number of values of n, such that the approximation furnished by the Fourier sum S_n will stand with respect to the best possible approximation in a ratio which is bounded.

In fact, since $f(x)$ has no critical point between two lines $y = \pm b$, which can be given, nor on those lines, the Fourier coefficients a_k and b_k are infinitely small with respect to e^{-kb}. However small $\epsilon > 0$ is given we can therefore assign a last subscript k such that

$$\sqrt{a_k^2 + b_k^2} > \epsilon\, e^{-kb},$$

and this value k will be as large as we like if we take ϵ sufficiently small.

The approximation furnished by the sum S_{k-1} is then less than the expression

$$\sum_{n=k}^{\infty} \sqrt{a_n^2 + b_n^2} < \sqrt{a_k^2 + b_k^2} \sum_{n=0}^{\infty} e^{-nb} = \frac{\sqrt{a_k^2 + b_k^2}}{1 - e^{-b}}$$

On the other hand, according to Bernstein's criterion, given at the end of the first lecture, the best possible approximation is greater than

$$\sqrt{\frac{1}{2} \sum_{n=k}^{\infty} (a_n^2 + b_n^2)} > \frac{1}{\sqrt{2}} \sqrt{a_k^2 + b_k^2}.$$

The ratio of these two approximations is then less than the fixed number $\sqrt{2/(1 - e^{-b})}$.

It is interesting to notice that this proof depends upon the order of convergence appropriate to analytic functions, and cannot be extended to other functions, such as the quasi-analytic functions with which we shall be concerned in the next lecture, even when their properties are very similar.

III

QUASI-ANALYTIC FUNCTIONS OF A REAL VARIABLE

1. *Borel's investigations and the concept of quasi-analytic function*

Analytic functions of a complex variable possess the property of being determined in their whole region of existence as soon as they are known in a region which is arbitrarily small. In fact, they are determined by their values and the values of all their successive derivatives at a single point z_0. This comes from the circumstance that they are developable in a Taylor's series in the neighborhood of the point, and the Taylor's development identifies, as we know, the analytic functions.

It was thought for a long time that the analytic functions were the only ones which were determined by their values and the values of their derivatives at a single point. It was Borel who first conceived and proved the existence of functions belonging to more general classes than that of analytic functions, which nevertheless possessed this same property; and he gave them the name *quasi-analytic*, by which they are now known.[1]

The functions which Borel considered are the *monogenic non-analytic* functions, defined in the complex plane, and on them he published a fundamental study in the collection

[1] *Les séries de fonctions analytiques et les fonctions quasi-analytiques*, Comptes Rendus, t. 154 (1912), p. 1491.

which bears his name.[1] But we shall not now take up the investigations of Borel, since for the present this study will not help us; and we shall only speak of the quasi-analytic functions considered in the real domain.

2. *Quasi-analytic functions of a real variable according to Denjoy's point of view*

This new method of treating quasi-analytic functions, which is much simpler than that of Borel, and which, as a matter of fact, is independent of that geometer's work, is due to Denjoy. Denjoy gave the exposition of his point of view and his results in a very remarkable article in the *Comptes Rendus*.[2] He considered quasi-analytic functions in a real interval, and showed that they could be defined by properties analogous to those which define analytic functions in such an interval, but which are merely *less restrictive*.

We shall first show that the functions which are analytic in a real interval, for instance, in the segment (0, 1), may be characterized by a condition of limitation imposed on their successive derivatives.

Let us suppose then that $f(z)$ is analytic on the segment $0 \leq x \leq 1$. This segment can be enclosed in a region D where $f(z)$ is still holomorphic and where its modulus does not exceed M. Let R be the shortest distance from the segment to the frontier of the region D. With any point X_0 of (0, 1) as center we can describe a circle C of radius R, interior to D. We apply accordingly the well-known formulae of Cauchy

[1] *Leçons sur les fonctions monogènes uniformes;* Paris, Gauthier-Villars (1917), edited by G. Julia. (See also Borel's lectures in The Book of the Opening of the Rice Institute (1912), vol. 2, pp. 399–430 [transl.]).

[2] Comptes Rendus, t. 173 (1921), p. 1329.

$$f(x_0) = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{z - z_0},\quad f^{(p)}(x_0) = \frac{p!}{2\pi i}\int_c \frac{f(z)dz}{(z - z_0)^{p+1}}$$

from which follow the formulae of limitation

$$|f| < M,\ |f^{(p)}| < \frac{p!M}{R^p}.$$

which are also well known.

In the last formula we may replace the $p!$ by its approximate value, given by Stirling's formula

$$\sqrt{2\pi p}\left(\frac{p}{e}\right)^p.$$

If now we notice that the expression

$$\frac{\sqrt[2p]{2\pi p M^2}}{e^R}$$

remains for all p less than a definite constant k, independent of p, we conclude, from the last inequality, the following one:

(C_0) $$\sqrt[p]{|f^{(p)}|} < kp.$$

Hence any function analytic on the segment (0, 1) satisfies the condition (C_0), for all p, with a proper value of the constant k. Conversely, a function which satisfies the condition (C_0) is analytic on the segment (0, 1), since this condition implies the convergence of the Taylor development about every point X_0 of that segment.

The condition (C_0), therefore, characterizes in the real domain the class of functions analytic on the segment (0, 1), and accordingly the functions of that class are completely determined by their values and the values of their successive derivatives at a point, let us say 0, of the segment.

Denjoy asked himself if it was not possible to enlarge the class (C_0) without making this property disappear. For this purpose he inserted factors which become infinitely

great with p into the second member of the condition (C_0) and in this way defined classes (C_1), (C_2), ... more and more extensive, and characterized by conditions

$$(C_1) \qquad \sqrt[p]{|f^{(p)}|} < kp \log p$$

$$(C_2) \qquad \sqrt[p]{|f^{(p)}|} < kp \log p \log\log p,$$

and so forth. He verified that the functions of these different classes were still completely determined by their initial values, and preserved for them the name *quasi-analytic* functions, proposed by Borel.

3. *The Denjoy-Carleman theorem*

These results were proved rigorously by Denjoy himself in the article mentioned, but he was led by induction to announce a further proposition, without however being able to prove it completely. He noticed that the reciprocals of the second members of the inequalities given above are themselves the general terms of divergent series, less and less rapidly divergent, and forming, in fact, a classic scale of such series. He was thus led to announce the following theorem:

Let $f(x)$ be a function which is indefinitely differentiable in the interval $(0, 1)$ and not analytic, and denote by M_0, M_1, M_2, ..., M_n, ... the absolute maxima of f, f', ..., $f^{(n)}$, ... in that interval; the function will be quasi-analytic, that is, completely determined in the whole interval by its value and the values of its derivatives in one point, if the series of positive terms

$$\frac{1}{M_1} + \frac{1}{\sqrt{M_2}} + \ldots + \frac{1}{\sqrt[n]{M_n}} + \ldots$$

is divergent.

This theorem was completely proved a fortnight later by Carleman,[1] and on that account it is called the *Theorem of Denjoy-Carleman*. I gave a complete proof of it in the lectures which it was my privilege to give in Paris in 1923.[2] But the complete demonstration of this theorem encounters many little difficulties of detail, which have only a secondary interest. Hence I am going to replace Denjoy's statement of the theorem by another which is simpler and avoids these secondary complications, and which brings out the general idea of the proof in its fine simplicity.

We shall then for the moment be satisfied with proving the following theorem, which defines a very general class of quasi-analytic functions.

4. *Theorem*

The class of functions which are indefinitely differentiable in the interval (0, 1) *and whose successive derivatives satisfy the condition of limitation*

$$(C) \qquad \sqrt[p]{|f^{(p)}|} < k\phi(p)$$

where k *is a constant and* $\phi(p)$ *a function which increases constantly and without limit, is a class of quasi-analytic functions if the series* $\sum \frac{1}{\phi(p)}$, *or* (*what amounts to the same thing*) *if the integral with infinite limit*

$$\int_0^\infty \frac{d\phi}{\phi(p)}$$

is divergent.

[1] Comptes Rendus, t. 174 (1922), p. 373. Another demonstration, of entirely different character, was given by Carleman, in a session of the Fifth Congress of Scandinavian Mathematicians, Helsingfors, 1923.

[2] *Four lectures on analytic functions of a real variable.* Bulletin de la Société Mathématique de France, No. Jubilaire, 1924.

A function of the class (C) is then completely determined in the whole interval (0, 1) by its value and the values of its successive derivatives at the point 0.

We can introduce several simplifications in the proof of the preceding theorem.

If two functions f_1 and f_2 take on the same initial values, as well as their derivatives, the initial values for their difference will vanish. In order that f_1 and f_2 be identical it is sufficient that their difference be identically zero. Now if the two functions f_1 and f_2 are of the class (C), their difference is of the same class. In order to prove the above theorem it will accordingly be sufficient to show that if a function of class (C) vanishes with all its derivatives at the point 0, the function vanishes identically in the interval (0, 1).

It suffices also to prove the theorem for a not negative function f, since we can replace f by f^2, which also belongs to the same class (C).

Finally, we may assume that $f(x)$ satisfies the symmetry condition

$$f(x) = f(1 - x),$$

since we can, if need be, replace $f(x)$ by the function

$$F(x) = f[4x(1 - x)]$$

which satisfies this condition and also belongs to the class (C). This fact is easily verified by calculating the n^{th} derivative.

It is therefore sufficient now to prove the following theorem:

5. *Theorem*

A function $f(x)$ of class (C), which is not negative and which vanishes with all its derivatives at both ends of the interval (0, 1), *vanishes throughout the whole of that interval.*

In order to prove the theorem we form, with Denjoy, the integral

$$F(z) = \int_0^1 f(x)e^{-zx}dx$$

considering z as a parameter. This integral defines an entire function $F(z)$ of the complex variable z, on the whole plane. This fact is immediate. Our theorem will then follow if we can prove that $F(z)$ is identically zero, for we shall have, for $z = 0$,

$$F(0) = \int_0^1 f(x)dx = 0,$$

and the function $f(x)$, being not negative, will be zero.

We see at once that the function $F(z)$ is bounded (since the same is true of e^{-zx}) to the right of the imaginary axis. If $F(z)$ were also bounded to the left of this axis, we could apply Liouville's theorem, and deduce that $F(z)$ is a constant, and therefore zero, since we know that $F(z)$ vanishes for z positively infinite. But we do not know how to make this proof. Denjoy evaded this difficulty by using ideas which testify to his great ingenuity, but succeed only with special forms of the function $\phi(p)$, introduced in the condition (C).

It was Carleman who found the real basis for the proof, and made known, at the same time, a very remarkable theorem of the theory of analytic functions. To this theorem I devoted the first of the lectures which I gave at Paris last year, and I refer you to that lecture for the proof.

Carleman's theorem is somewhat analogous to the theorem of Liouville, but whereas Liouville's theorem refers to a function considered over the whole plane, that of Carleman concerns the half plane. It says that a function $F(z)$, which is bounded in the half plane to the right of the imaginary axis and approaches 0 as z becomes infinite along the

imaginary axis in either direction, will necessarily reduce to zero if the rapidity of decrease of the function exceeds a certain limit.

We can easily state such a limit.

Let $z = re^{\theta i}$. According to our hypothesis that F approaches zero, we can write

$$|F(re^{\pm\frac{\pi i}{2}})| < Ae^{-\psi(r)},$$

with $\psi(r)$ an increasing function of r which becomes infinite with r, and the more rapidly according as F tends more rapidly to zero.

This is now Carleman's theorem:

If $\psi(r)$ increases sufficiently fast so that the integral to infinity

$$\int^{\infty} \frac{\psi(r)dr}{r^2}$$

is divergent, we shall have identically $F(z) = 0$.

We are thus led, in order to apply this theorem to Denjoy's integral $F(z)$, to investigate how $F(z)$ decreases as z tends to infinity along the imaginary axis. For this purpose we make p consecutive integrations by parts, and observe that since $f(z)$ and its derivatives vanish at the two ends of the interval all the terms at these extremities are zero. Hence

$$F(z) = \int_0^1 \frac{f^{(p)}(x)}{z^p} e^{-zx}\, dx.$$

From this we have, by condition (C),

$$|F(re^{\pm\frac{\pi i}{2}})| < \left[\frac{k\phi(p)}{r}\right]^p.$$

Let us fix the integer p by the conditions

$$\phi(p) \leq \frac{r}{ke} < \phi(p+1)$$

whence, if we designate by ψ the function inverse to ϕ,

$$p \leq \psi\left(\frac{r}{ke}\right) < p+1.$$

We shall have *a fortiori*

$$| F(re^{\pm\frac{\pi i}{2}}) | < Ce^{-\psi\left(\frac{r}{ke}\right)}.$$

In this way the condition of limitation on the function F is obtained by means of the inverse of the function ϕ.

In order that F be identically zero, it is therefore sufficient, in accordance with the above theorem from the theory of analytic functions, that the integral with infinite limit

$$\int^{\infty} \frac{\psi\left(\frac{r}{ke}\right)}{r^2}\, dr \quad \text{or (what amounts to the same thing)} \quad \int^{\infty} \frac{\psi(r)}{r^2}\, dr$$

be divergent.

But this fact is precisely the consequence of condition (C), because the integral of condition (C) is reduced to the above integral by an integration by parts; and both integrals are divergent at the same time. We have, in fact, if n is the inverse of ϕ,

$$\int \frac{dn}{\phi(n)} = \frac{n}{\phi(n)} + \int \frac{n d\phi}{\phi^2} = \frac{n}{\phi(n)} + \int \frac{\psi(\phi)}{\phi^2}\, d\phi.$$

The function $f(x)$, by hupothesis, fails to be analytic; hence $\phi(n)$ surely exceeds n for certain properly chosen values of n, sufficiently great, and the last integral (in which ψ is positive) is infinite whenever the first is. It is clear, conversely, that under the same condition, the divergence of the last integral implies the divergence of the first integral, in the preceding relations.

The theorem is thus fully established.

6. *Identification of quasi-analytic functions by the order of decrease of the coefficients of their trigonometric development*

The investigations of Denjoy and Carleman are exceedingly interesting, but they leave a gap in the theory. In fact, they characterize the quasi-analytic functions by properties which cannot immediately be tested in terms of their analytic expression, and moreover do not lead to any practical method of analytic representation for those functions. This is true to such an extent, that one would be at a loss to define explicitly particular quasi-analytic functions, and one might even place their existence in doubt. I have shown, by applying to these special functions the methods which I utilized in the study of the approximation of functions, that the difficulties which I have just mentioned disappear if we consider the development of quasi-analytic functions in trigonometric series.

It is of course understood that this trigonometric representation is applicable only in the case of periodic functions. We shall consider therefore only a function $f(x)$ with period 2π. This restriction does not interfere with the generality of the theory, since it is always satisfied if we replace the variable by $\cos x$. This substitution, performed on a function of class (C) does not alter, in general, the condition of limitation for the derivatives appropriate to this class, but we shall not stop to prove the statement. There is nothing, in fact, to prevent us from studying the properties of the function when the substitution has been made.

In order to simplify the writing we shall consider an even function. The above mentioned substitution always leads to such a function. But we notice also that any function is the sum of an even and an odd function, and

that the derivative of an odd function is an even function. Hence in any case, matters come back to the even function.

Let $f(x)$ then be an even function, of period 2π and indefinitely differentiable. It is expressible in a trigonometric series, indefinitely differentiable, of the form

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty a_k \cos kx.$$

We want to show that the various classes of quasi-analytic functions can be characterized by the law of limitation imposed on their Fourier coefficients a_k as well as by that imposed on their successive derivatives. This is the consequence of the formulae which I established in the preceding lecture.

We have seen that the condition

$$\sqrt[n]{|f^{(n)}|} < \phi(n)$$

implies as a consequence the condition

$$|a_k| < Ae^{-\psi\left(\frac{k}{e}\right)} \qquad (A \text{ const.}),$$

where the function ψ is the inverse of the function ϕ. Also, we saw that conversely the condition

$$|a_k| < Ae^{-\psi(k)}$$

implies as a consequence the condition

$$\sqrt[n]{|f^{(n)}|} < B\phi\left(\frac{2n}{\alpha}\right), \qquad (B \text{ const.}),$$

with the proviso that we can choose α, positive, so that the function $\psi(x)/x^\alpha$ is continually increasing. Finally, we know that in the case of a function $f(x)$ which is not analytic, the integrals with infinite limit

$$\int^\infty \frac{dn}{\phi(n)} \qquad \int^\infty \frac{\psi(k)\,dk}{k^2}$$

are both divergent or both convergent.

From this we have the following conclusion:

The class (C) of quasi-analytic functions which we have already defined may equally well be characterized by the condition of limitation

$$(C) \qquad |a_k| < Ae^{-\psi(k)} \qquad (k = 0, 1, 2, \ldots)$$

imposed on its Fourier coefficients, subject to the hypotheses: (1) that there exist a positive α such that the function $\psi(k)/k^\alpha$ shall be non-decreasing, and (2) that the integral

$$\int^{\infty} \frac{\psi(k)\, dk}{k^2}$$

shall be divergent.

Of course, these conditions are not assumed to hold in the interval (0, 1) as before, but in the interval of a complete period (of amplitude 2π), and, therefore, for all values of x (on account of the periodicity).

We can apply the above principle to the definition of the classes (C_1), (C_2) ..., which were considered originally by Denjoy, and which enter as particular cases in the above class (C), by giving to ψ the particular evaluations:

$$(C_1) \qquad \psi(k) = \frac{k}{\log k}$$

$$(C_2) \qquad \psi(k) = \frac{k}{\log k \ \log \log k}$$

and so on.

It is now possible to write down at once a quasi-analytic function which belongs to one of these classes. Thus, for instance, the function

$$f(x) = \Sigma\, e^{\frac{k}{\log k}} \cos kx$$

is a quasi-analytic function which belongs to the first Denjoy class (C_1).

By utilizing the results already obtained, it would not be difficult to characterize certain classes of quasi-analytic

functions by the order of approximation which they admit by means of a finite expression of order n. This is not the method of definition which seems most natural to us, but we may notice here that Bernstein has taken this direction of approach and obtained quite recently some very interesting results.[1]

7. *Calculation of a quasi-analytic function for which the initial values are given*[2]

We take up the problem of determining a periodic quasi-analytic function, assumed to exist, by its initial value and the initial values of its derivatives for $x = 0$. It suffices evidently to be able to solve the problem for an even function, for as we have said, every function is the sum of an even and an odd function. The derivatives of odd order of the former vanish at the origin, and the same applies to the derivatives of even order of the latter. The determination of the two functions can thus be done independently. Finally, the determination of an odd function reduces to the calculation of its derivative, which is an even function, — a fact which justifies our statement.

The problem then is to determine the even quasi-analytic function

$$f(x) = \Sigma a_r \cos rx$$

on the basis of the following circumstances:

(1) We assume that the function exists and belongs to the class defined by the condition

$$(1) \qquad |a_r| < Ae^{-\psi(r)}$$

[1] *Sur les fonctions quasi-analytiques de M. Carleman*, Comptes Rendus, t. 179 (1924), p. 743.

[2] The solution of this problem was given by Carleman, by developing the function in a series of polynomials [Comptes Rendus, t. 176 (1923), p. 64]. The solutions which we give here are much simpler.

where A is a given constant and $\psi(r)$ a given function which becomes infinite with r in such a way as to insure the infinite differentiability and the quasi-analyticity of the function.

(2) We assume that the function and its derivatives of even order take on the initial values $C_0, C_2, \ldots, C_{2i}, \ldots$

In accordance with this, we have the system of equations

$$(2) \qquad C_{2i} = \sum_{r=0}^{\infty} (-1)^i r^{2i} a_r \qquad (r = 0, 1, 2, \ldots),$$

to determine the Fourier coefficients.

We shall first show that the solution of the problem reduces to the successive solution of linear inequalities. The calculation of $f(x)$ depends, in fact, on a passage to the limit which rests on the indefinite repetition of the operation which we are going to describe once for all.

We assign two positive numbers ϵ and n, the first as small, the second, which is an integer, as large as we please. Consider the first n equations (2); without knowing the a_r, but making use of the conditions (1) which determine a definite order of convergence of the series, we can take in each sum of the second member of the first n equations (2), a sufficient number of terms so that the remainders of these n series will be $< \epsilon$ in absolute value. Suppose that for this purpose, to make our notions precise, we take the same number, $p + 1$, of terms in each equation, although this is not necessary. We shall have the n inequalities:

$$(3) \quad \left| C_{2i} - \sum_{r=0}^{p} (-1)^i r^{2i} a_r \right| < \epsilon \qquad [i = 0, 1, 2, \ldots, (n-1)].$$

The function $f(x)$ is assumed to exist, so that its Fourier coefficients $a_o, \ldots, a_p$ satisfy the $(p + 1)$ above mentioned inequalities (1) and the n inequalities (3). Hence if we consider the coefficients a_r as unknown these inequalities form a system of $n + p + 1$ linear inequalities among the

$a_0, a_1, \ldots, a_p$ which we are sure have solutions. The determination of a particular system of solutions is a problem which may be long, but the solution of it is elementary. Let us find then, by any method we like, a particular system of solutions $a'_0, a'_1, \ldots, a'_p$. We form thus the function

(4) $\phi(x) = a'_0 + a'_1 \cos x + a'_2 \cos 2x + \ldots + a'_p \cos px$

which may be considered as an approximate solution of the problem.

This is the method of calculation which must be repeated an infinite number of times, and leads to the solution of the problem.

We take an infinite sequence of pairs of values of ϵ, p. Let $(\epsilon_1, p_1), (\epsilon_2, p_2), \ldots (\epsilon_n, p_n), \ldots$ be these various pairs, ϵ_n approaching zero and p_n becoming infinite. To these successive pairs correspond successive functions $\phi_1, \phi_2, \ldots$ $\phi_n, \ldots$ of the form (4). I say now that this sequence of functions ϕ_n is convergent, and that ϕ_n converges to the desired quasi-analytic function $f(x)$, as n becomes infinite.

In fact, all the functions ϕ_n are of the form (4) where the coefficients $a'_0, a'_1, a'_2, \ldots$ are subject to the conditions (1) and are consequently enclosed in intervals which insure the absolute and uniform convergence of the series and the quasi-analyticity of the sum function. The sets formed by the various values of the coefficients $a'_0, a'_1, \ldots$ respectively are bounded and consequently have at least one limit point. We can take from the sequence of the ϕ_n a first sequence where a'_0 converges, from this first a second sequence where a'_1 converges, and so on. In this way we determine in the limit a quasi-analytic function which admits the initial values C_{2i}, and this function is nothing else than the $f(x)$ desired, since $f(x)$ is the only one which can take on these initial values.

But even the complete sequence $\phi_1, \phi_2, \ldots, \phi_n, \ldots$

converges to $f(x)$. For if this sequence failed to converge there would be at least two limit points for the set of values given to at least one of the coefficients a', say a'_0. In this case we could, following the above method, determine two quasi-analytic functions with the same initial values but different coefficients a'_0. Hence the functions would be different, which is impossible.

8. *Modification of the above method of solution*

The method of solution which I have just developed is the one which is probably closest to the nature of the problem and clarifies it most. But we may look for a more elegant procedure, and, in fact, imitate the one already invented by Carleman.[1] This method avoids the resolution of inequalities by means of an infinite process depending on the method of least squares. This is the way I followed in my lectures last year at Paris, but the method I used then may again be modified and put in a more satisfactory form, which I will now proceed to outline.

Take again two numbers ϵ and n, the first as small, and the second integral and as large as we please. In virtue of the assumed conditions, namely

$$|a_r| < Ae^{-\psi(r)}, \tag{1}$$

we can choose the integer p great enough so that the coefficients $a_0, a_1, \ldots, a_p$ of the quasi-analytic function desired will satisfy the n conditions:

$$\left(C_{2i} - \sum_0^p r^{2i} a_r\right)^2 < \frac{\epsilon}{n}, \quad (i = 0, 1, \ldots n-1). \tag{2}$$

We write down the quadratic form in the $p+1$ variables $y_0, y_1, y_2, \ldots y_n$,

$$F(y) = \sum_{i=0}^{n-1} \left(C_{2i} - \sum_0^p r^{2i} y_i\right)^2 + \epsilon \sum_0^p y_r^2 e^{\psi(r)}. \tag{3}$$

[1] Comptes Rendus, t. 176 (1923), loc. cit.

This is a definite positive form. Let us find its minimum. The values $a'_0, a'_1, \ldots, a'_p$ of the y which give a minimum are obtained by solving a system of linear equations.

This minimum of $F(y)$ is not greater than the value of $F(y)$ when we put for $y_0, y_1, y_2, \ldots$ the coefficients $a_0, a_1, a_2, \ldots$ of the desired function $f(x)$, hence less than the quantity

$$n\frac{\epsilon}{n} + \epsilon A^2 \sum_{r=0}^{p} e^{-\psi(r)} = (1 + A^2 B)\epsilon$$

where we designate by B the value of the convergent series of exponentials.

It follows therefore that the values $a'_0, a'_1, \ldots a'_p$ which make $F(y)$ a minimum make each term of the sum (3) less than that limit. Hence we have the conditions

$$\left(C_{2i} - \sum_{0}^{p} r^{2i} a'_r\right)^2 < (1 + A^2 B)\epsilon \tag{4}$$

$$a'^2_r e^{\psi(r)} < (1 + A^2 B), \text{ whence } a'_r < \sqrt{1 + A^2 B}\, e^{-\frac{\psi(r)}{2}}. \tag{5}$$

By means of the above process, to each system ϵ, p, we make correspond a function

$$\phi(x) = a'_0 + a'_1 \cos x + \ldots + a'_p \cos px$$

whose coefficients satisfy the conditions (4) and (5). It is an approximate solution in the same sense as in the previous method. As ϵ approaches 0 and p becomes infinite this approximate solution approaches the desired quasi-analytic function, since the conditions (5) insure the convergence of the process and the quasi-analyticity of the limit as in the previous method.

9. *On the existence of a quasi-analytic function which admits the initial values*

The success of the methods which we have just explained is based on the hypothesis that the quasi-analytic function $f(x)$ exists which takes on the given initial values. What

happens if, given the initial values and the class of the quasi-analytic function, we are not sure of the existence of a solution?

Suppose we apply the first method, which consists in solving systems of inequalities. Two cases are possible.

One alternative is that at a certain stage we find ourselves stopped in the sequence of operations, by encountering a system of incompatible inequalities. In this case we know that the problem is impossible and that there is no quasi-analytic function of the proposed class which with its derivatives takes on the initial values.

The other alternative is that after having carried the calculation more or less far, we stop before having encountered an incompatibility. In this second case we shall have obtained a more or less approximate solution of the problem, but we will not be sure of the existence of an exact solution, since it is always possible that we may meet an incompatibility by pushing the calculation further.

The question of finding a criterion of consistence for the initial values of a quasi-analytic function, of a certain class, and its derivatives, is thus before us. But it seems exceedingly difficult to solve. We shall not undertake it here.

Nevertheless, in a Note which we append to this lecture we shall explain some results with regard to a related but much simpler problem, that of constructing a function, quasi-analytic or not, taking on with its successive derivatives a system of given initial values.

NOTE

ON THE DETERMINATION OF PERIODIC FUNCTIONS BY MEANS OF THEIR INITIAL VALUES

1. *Purpose of this Note*

Consider a trigonometric series containing (for greater simplicity) only cosine terms,

$$\Sigma a_n \cos nx.$$

We know, by what has preceded, that if the coefficients a_n satisfy a law of decrease which is sufficiently rapid, the sum of the series is an analytic function of x. Suppose that we assign *a priori* the initial values $C_0, C_2, C_4, \ldots$ of a function $f(x)$ and its successive even derivatives for $x = 0$; it will in general be impossible to satisfy these conditions with a periodic function whose Fourier coefficients decrease rapidly enough to make it analytic. In fact, an analytic function is defined at once in terms of the initial values by means of its Taylor development, and this power series represents a periodic function only exceptionally.

On the other hand, if we do not assign a special law of decrease for the Fourier coefficients, it is always possible to construct a periodic function which, for $x = 0$, takes on, with its successive derivatives of even order, the sequence of given values $C_0, C_2, \ldots$, whatever they may be.

It is our plan to prove this statement, and to outline a method of constructing such a function. We shall add some remarks which relate naturally to this question.

2. *Theorem I*

Whatever initial values $C_0, C_2, \ldots, C_{2n}, \ldots$ *are given, it is always possible to form an even periodic function* $f(x)$, *such that the differences*

$$f^{(2n)}(0) - C_{2n} \quad (n = 1, 2, 3, \ldots)$$

are bounded for all n.

Let $m_1, m_2, \ldots m_n, \ldots$ be an increasing sequence of integers and $a_1, a_2, \ldots a_n, \ldots$ an arbitrary sequence of coefficients. We wish to determine them one after another. We write

$$f(x) = C_0 + \sum_{n=1}^{\infty} a_n \frac{\cos m_n x}{m_n^{2n-1}}.$$

From this, we have

$$-f''(0) = m_1 a_1 + \frac{a_2}{m_2} + \frac{a_3}{m_3^3} + \ldots$$

$$f^{\text{iv}}(0) = m_1^3 a_1 + m_2 a_2 + \frac{a_3}{m_3} + \ldots$$

and so on.

Let us form the system of successive equations:

$$\begin{aligned} -C_2 &= m_1 a_1 \\ C_4 &= m_1^3 a_1 + m_2 a_2 \\ -C_6 &= m_1^5 a_1 + m_2^3 a_2 + m_3 a_3, \end{aligned}$$

and so forth. This is a recurrent system which determines successively the products $m_1 a_1, m_2 a_2, \ldots m_n a_n, \ldots$. When these products are known it is obvious that we can always take for the first factors $m_1, m_2, \ldots m_n, \ldots$ an increasing sequence of integers, and moreover one that increases fast enough so that the second factors $a_1, a_2, \ldots a_n, \ldots$ are all contained between -1 and $+1$. The function $f(x)$ is then completely determined by the above trigonometric series, and it is evident that this series satisfies the conditions of the theorem.

Not only are the differences $f^{(2n)}(0) - C_{2n}$ bounded for all n, but in our case they also approach 0 as n becomes infinite.

The preceding theorem reduces the determination of a function which is even and periodic and with its successive derivatives takes on a sequence of given initial values to the same problem where the initial values $C_0, C_2, \ldots, C_{2n}, \ldots$ are bounded. This problem will be solved in the proof of the following theorem.

3. *Theorem II*

Let $m_1, m_2, \ldots m_n, \ldots$ be a sequence of positive integers, increasing rapidly enough so that for all n,

$$\frac{m_{n+1}}{m_n} > \lambda > 1$$

where λ is a fixed positive number. Let then $C_0, C_2, \ldots, C_{2k}, \ldots$ be a sequence of bounded numbers. It is always possible to determine the coefficients of the trigonometric series $f(x) = \alpha_1 \cos m_1 x + \alpha_2 \cos m_2 x + \ldots + \alpha_n \cos m_n x + \ldots$ in such a way that this series and all its derivatives will converge, and the sum function and its derivatives of even order will take on respectively the given values $C_0, C_2, \ldots$ for $x = 0$.

We determine first n coefficients $a_1, a_2, \ldots a_n$, by the system of n equations

$$(1) \quad \begin{cases} C_0 = a_1 + a_2 + \ldots + a_n \\ -C_2 = m_1^2 a_1 + m_2^2 a_2 + \ldots + m_n^2 a_n \\ \pm C_{2n-2} = m_1^{2n-2} a_1 + m_2^{2n-2} a_2 + \ldots + m_n^{2n-2} a_n \end{cases}$$

and write

$$f_n(x) = a_1 \cos m_1 x + a_2 \cos m_2 x + \ldots + a_n \cos m_n x.$$

Evidently, it suffices to show that as n becomes infinite, $f_n(x)$ approaches a limit function $f(x)$ which satisfies the theorem. For this purpose, we shall show that $a_1, a_2,$

$a_3, \ldots$ approach determinate limits $\alpha_1, \alpha_2, \alpha_3, \ldots$ respectively, and are contained respectively within limits (whatever the value of n) which insure the convergence of the sum (which changes with n)

$$a_1 \cos m_1 x + a_2 \cos m_2 x + \ldots, \tag{2}$$

and its successive derivatives.

We solve first the system (1). Its determinant Δ is a Vandermonde determinant which may be resolved into a product of factors

$$\Delta = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ m_1^2 & m_2^2 & \ldots & m_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ m_1^{2n-2} & m_2^{2n-2} & \ldots & m_n^{2n-2} \end{vmatrix}$$

$$= (m_2^2 - m_1^2)(m_3^2 - m_1^2)\ldots(m_n^2 - m_1^2)$$
$$(m_3^2 - m_2^2)\ldots(m_n^2 - m_2^2)$$
$$\ldots \quad \ldots$$
$$(m_n^2 - m^2{}_{n-1}).$$

Denote by $\Delta_1(z)$ the polynomial in z obtained by replacing the letter m_1 in Δ by z; we shall have

$$\frac{\Delta_1(z)}{\Delta} = \frac{(m_2^2 - z^2)(m_3^2 - z^2)\ldots(m_n^2 - z^2)}{(m_2^2 - m_1^2)(m_3^2 - m_1^2)\ldots(m_n^2 - m_1^2)}.$$

According to the theory of equations, the value of the unknown a_1 is equal to a fraction of which the denominator is the determinant Δ and the numerator is obtained by replacing in this determinant the elements of the first column $1, m_1^2, \ldots$ by $C_0, -C_2, \ldots$. Evidently we arrive at the same result by ordering the numerator of the fraction written above according to powers of z^2 and replacing $z^0, z^2, \ldots z^{2n-2}$ by $C_0, -C_2, \ldots, \pm C_{2n-2}$ respectively. The result of this substitution may be given explicitly with the help of a complex definite integral. This will now be shown.

Let M be a positive quantity greater than the absolute

values of all the quantities C_{2k}. We describe a circle (C) of radius M around the origin of the z's. Outside and on the boundary of this circle we may define

$$\psi(z) = \frac{C_0}{z} - \frac{C_2}{z^3} + \frac{C_4}{z^5} - \cdots,$$

since this series is convergent in that region. If now we integrate along the circle (C) we have (according to the theory of residues)

$$a_1 = \frac{1}{2\pi i}\int_c \frac{\Delta_1(z)}{\Delta}\,\psi(z)dz.$$

In order to study this expression and its analogs, we write

$$\phi_n(z) = \left(1 - \frac{z}{m_1^2}\right)\left(1 - \frac{z}{m_2^2}\right)\cdots\left(1 - \frac{z}{m_n^2}\right);$$

whence we have, without difficulty, ϕ' being the derivative of ϕ,

$$\frac{\Delta_1(z)}{\Delta} = \frac{\phi_n(z^2)}{(z^2 - m_1^2)\phi'_n(m_1^2)},$$

and accordingly

$$a_1 = \frac{1}{2\pi i}\int_c \frac{\phi_n(z^2)\,\psi(z)dz}{(z^2 - m_1^2)\phi_n'(m_1^2)}.$$

The corresponding values of a_2, a_3, ... are obtained by a simple permutation of the letters. Permuting m_1 and m_k, we have

$$a_k = \frac{1}{2\pi i}\int_c \frac{\phi_n(z^2)\,\psi(z)dz}{(z^2 - m_k^2)\,\phi_n'(m_k^2)}. \tag{3}$$

The denominators, only, depend on k.

As n grows indefinitely great, the product denoted by $\phi_n(z^2)$ acquires new factors and is developed as a product with an infinite number of factors, which we may denote by $\phi(z)$,

$$\phi(z) = \left(1 - \frac{z}{m_1^2}\right)\left(1 - \frac{z}{m_2^2}\right)\cdots.$$

Since this product converges, a_k approaches a definite limit α_k which is expressed by the integral

$$\alpha_k = \frac{1}{2\pi i}\int_c \frac{\phi(z^2)\psi(z)dz}{(z^2 - m_k^2)\phi'(m_k^2)}.$$

In order to complete the demonstration it remains to show that as n varies the sum (2) and its derivatives are uniformly convergent. For this purpose we shall seek an upper bound of the $|\ a_k\ |$, which, when put in place of a_k in this sum, will insure the absolute convergence of this sum (extended to infinity) and its successive derivatives.

We return to the expression (3) for the a_k. In order to obtain an upper bound for this expression we must look for a lower bound of the absolute value of its denominator, or (what amounts to the same thing) of

$$m_k^2\phi_n'(m_k^2).$$

This expression, like $\phi(z^2)$, is a product of factors. We divide them into two groups. The first group, formed from the first $k-1$ factors, has the form

$$\left(\frac{m_k^2}{m_1^2} - 1\right)\left(\frac{m_k^2}{m_2^2} - 1\right)\cdots\left(\frac{m_k^2}{m_{k-1}^2} - 1\right).$$

It can be put in the form

$$\left(\frac{m_k^{k-1}}{m_1 m_2 \cdots m_{k-1}}\right)^2\left(1 - \frac{m_{k-1}^2}{m_k^2}\right)\left(1 - \frac{m_{k-2}^2}{m_k^2}\right)\cdots,$$

and consequently exceeds in value the expression

$$\left(\frac{m_k^{k-1}}{m_1 m_2 \cdots m_{k-1}}\right)^2\left(1 - \frac{1}{\lambda^2}\right)\left(1 - \frac{1}{\lambda^4}\right)\cdots.$$

The second group, made up of the remaining factors, has the form

$$\left(1 - \frac{m_k^2}{m_{k+1}^2}\right)\left(1 - \frac{m_k^2}{m_{k+2}^2}\right)\cdots\left(1 - \frac{m_k^2}{m_n^2}\right),$$

and exceeds the definite quantity defined by the convergent infinite product

$$\left(1-\frac{1}{\lambda^2}\right)\left(1-\frac{1}{\lambda^4}\right)\ldots.$$

From this it follows that we can assign a constant h, and then a second constant h' (independent of k and n) so that we have

$$|\, m_k{}^2\phi_n'(m_k{}^2)\,| > h\left(\frac{m_k{}^{k-1}}{m_1 m_2 \ldots m_{k-1}}\right)^2$$

and consequently

$$|\, a_k\,| < h'\left(\frac{m_1 m_2 \ldots m_{k-1}}{m_k{}^{k-1}}\right)^2.$$

This upper bound (for k infinite) is an infinitesimal of higher order than any negative power of the m_k; it guarantees therefore the conditions of convergence which we have demanded.

We notice finally that the α_k, which are limits of the a_k, have the same upper bounds.

Theorem II is thus completely proved.

4. *On the law of limitation for the Fourier coefficients*

We shall make the calculation of the last section more definite in a particular case. Let the numbers $m_1, m_2, \ldots, m_k, \ldots$ be successive powers of the same integer $\lambda > 1$.

We consider in this case the trigonometric series

$f(x) = \alpha_1 \cos \lambda x + \alpha_2 \cos \lambda^2 x + \ldots + \alpha_k \cos \lambda^k x + \ldots$

in which the upper bound of the α_k, found at the end of the preceding section, takes the form

$$|\, \alpha_k\,| < h'\left(\frac{\lambda\cdot\lambda^2\ldots\lambda^{k-1}}{\lambda^{k(k-1)}}\right) = \frac{h'}{\lambda^{k(k-1)}} = h'e^{-k(k-1)\log\lambda}.$$

The series which we have just considered comes under the general type of trigonometric series

$$a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx + \ldots,$$

where all the coefficients vanish except when $n = \lambda^k$, in which case $a_n = \alpha_k$. Let us express the law of limitation of the a_n as a function of n. We have $k = \log n / \log \lambda$; consequently

$$|a_n| < h' e^{-\log n \left(\frac{\log n}{\log \lambda} - 1\right)}.$$

We can accordingly fix two constants A and a so that we shall have for any n

$$|a_n| < A e^{-a(\log n)^2}. \tag{4}$$

This gives us the following theorem:

There are always an infinity of periodic functions which with their successive derivatives take on for $x = 0$ an arbitrarily given bounded sequence of values, and such that their Fourier coefficients a_n obey a law of limitation of the form (4).

5. *On certain classes of functions determined by the system of initial values*

It is possible to define, in terms of the form of their trigonometric development, certain classes of functions which are neither analytic nor quasi-analytic, and yet such that a function of the class is determined by its value and that of its successive derivatives at $x = 0$.

We have, in fact, the following theorem:

Let $m_1, m_2, \ldots m_n \ldots$ be a sequence of positive integers, increasing sufficiently fast so that the quotient

$$\frac{m_1 m_2 \ldots m_n}{m_{n+1}}$$

approaches 0 *as n becomes infinite. The functions which are representable by the trigonometric series*

$$a_1 \cos m_1 x + a_2 \cos m_2 x + \ldots + a_n \cos m_n x + \ldots$$

with the condition of limitation

$$|a_n| < h' \left(\frac{m_1 m_2 \ldots m_{n-1}}{m_n{}^{n-1}} \right)^2 \qquad (h' \text{ const.})$$

form a class in which every function is determined by its value and those of its derivatives at $x = 0$. *Moreover we find that these values remain arbitrary, provided they are bounded.*

We notice first that the sum of two functions of the class considered belongs to the same class. With this fact, it is enough to prove that a function of this class vanishes identically if all its initial values are zero.

Suppose then that all the initial values are zero. The n first coefficients $a_1, a_2, \ldots, a_n$ must satisfy the equations (1) of section 3, provided we consider there the first members $C_0, -C_2, \ldots$ as representing, with opposite signs, the portions of infinite series discarded from the second members (or the remainders of those series). The calculation made in section 3 shows (on letting n become infinite) that $a_1, a_2, \ldots$ are all zero, if these remainders themselves approach zero.

The two conditions in the statement of the theorem allow us to satisfy ourselves easily that the first term of the remainder in the last equation of the system approaches zero. The whole remainder also approaches zero on account of the rapidity of decrease of the coefficients. All the remainders approach zero *a fortiori* in the other equations of the system. The theorem is therefore proved.

For example, the class of functions which may be given by trigonometric series

$$a_1 \cos x + a_2 \cos 2^2 x + a_3 \cos 3^3 x + \ldots + a_n \cos n^n x + \ldots$$

with the limitation condition

$$|a_n| < h \left(\frac{1 \cdot 2^2 \cdot 3^3 \ldots (n-1)^{n-1}}{n^{(n-1)n}} \right)^2 \qquad (h \text{ const.})$$

is of the preceding kind: a function of this class is completely determined by its value and that of its successive derivatives for $x = 0$.

The functions of this class are neither analytic nor quasi-analytic.

Charles de la Vallée Poussin.

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A COURSE OF THREE LECTURES ON PHOTOCHEMISTRY DELIVERED BY E. C. C. BALY, C.B.E., M.Sc., F.R.S., Grant Professor of Inorganic Chemistry in the University of Liverpool, and Fellow of University College, London, in the Physics Amphitheatre of the Rice Institute, October 23, 24, and 25, 1924.

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